J. Math. Anal. Appl. 415 (2014) 373–393

Contents lists available at ScienceDirect

Journal of Mathematical Analysis and Applications

www.elsevier.com/locate/jmaa

A fast alternating minimization algorithm for total variation deblurring without boundary artifacts

Zheng-Jian Bai [a,1], Daniele Cassani [b,*], Marco Donatelli [b,2], Stefano Serra-Capizzano [b,2]

[a] *School of Mathematical Sciences, Xiamen University, Xiamen 361005, PR China*
[b] *Dipartimento di Scienza e Alta Tecnologia, Università degli Studi dell'Insubria, Via Valleggio 11, 22100 Como, Italy*

ARTICLE INFO

Article history:
Received 31 July 2013
Available online 24 January 2014
Submitted by V. Radulescu

In memory of Alfredo Lorenzi

Keywords:
Image deblurring
Reflective and anti-reflective boundary conditions
Total variation
Variational methods

ABSTRACT

Recently, a fast alternating minimization algorithm for total variation image deblurring (FTVd) has been presented by Wang, Yang, Yin, and Zhang (2008) [32]. The method in a nutshell consists of a discrete Fourier transform-based alternating minimization algorithm with periodic boundary conditions and in which two fast Fourier transforms (FFTs) are required per iteration. In this paper, we propose an alternating minimization algorithm for the continuous version of the total variation image deblurring problem. We establish convergence of the proposed continuous alternating minimization algorithm. The continuous setting is very useful to have a unifying representation of the algorithm, independently of the discrete approximation of the deconvolution problem, in particular concerning the strategies for dealing with boundary artifacts. Indeed, an accurate restoration of blurred and noisy images requires a proper treatment of the boundary. A discrete version of our continuous alternating minimization algorithm is obtained following two different strategies: the imposition of appropriate boundary conditions and the enlargement of the domain. The first one is computationally useful in the case of a symmetric blur, while the second one can be efficiently applied for a nonsymmetric blur. Numerical tests show that our algorithm generates higher quality images in comparable running times with respect to the Fast Total Variation deconvolution algorithm.

1. Introduction

The bond between mathematics and visual observations has deep roots, down to the very beginning of science and technology. Nowadays, image processing enters many different areas of sciences such as engineering, biology, medical sciences, breaking through everyday life. The basic problem of image restoration,

* Corresponding author.
E-mail addresses: zjbai@xmu.edu.cn (Z.-J. Bai), daniele.cassani@uninsubria.it (D. Cassani), marco.donatelli@uninsubria.it (M. Donatelli), stefano.serrac@uninsubria.it (S. Serra-Capizzano).

[1] Research partially supported by the National Natural Science Foundation of China grant 11271308, NCET, and Internationalization Grant of U. Insubria 2008.
[2] Work partially supported by MIUR, grant number 20083KLJEZ.

0022-247X/$ – see front matter
http://dx.doi.org/10.1016/j.jmaa.2014.01.061

once any kind of corruption has occurred, has been tackled with the aid of computer technology, whose development from one side relies on the implementation of old mathematical tools, such as classical Fourier Analysis, and on the other side promotes new mathematical results and throws light on new theoretical as well as applied challenges.

Here we consider the Total Variation (TV) image deblurring problem by minimizing the following energy functional

$$E(u) := \frac{\alpha}{2}\|\mathcal{H}u - f\|^2_{L^2(\Omega)} + \int_{\Omega} |\nabla u|\, dx, \tag{1}$$

where $\alpha > 0$ is a fidelity parameter, Ω is an open rectangular domain in $\mathbb{R}^2$, $\mathcal{H}$ is a given linear blurring operator, $f : \Omega \to \mathbb{R}$ is the observed image in $L^2(\Omega)$, u is the unknown image to restore, and $|\cdot|$ denotes the Euclidean norm [31,9]. The second term in (1) is the total variation of u and represents the energy obstruction to high frequency noise affecting the original image which is out of reach to human eyes and thus made unfavorable. We merely mention that functionals of this type arise in different topics such as Cheeger's sets in differential geometry [22], degenerate singular diffusion PDE and the 1-Laplacian [18], elastic plastic problems [29].

The blurring model is assumed to be space-invariant, namely the Point Spread Function (PSF) is represented by a specific real bivariate function $h(x - y)$, $x, y \in \Omega$, for some univariate function $h(\cdot)$ [21]. According to the linear modeling proposed in the literature [20], the observed image f and the original image u are described by the relation

$$f(x) = \mathcal{H}u(x) + \eta(x) := \int_{\Omega} h(x - s)u(s)\, ds + \eta(x), \quad x \in \Omega, \tag{2}$$

where the kernel h is the PSF and η is the noise.

A first step in the classical approach is to discretize (2) as follows

$$\mathbf{f} = A\mathbf{u} + \boldsymbol{\eta}, \tag{3}$$

with $A \in \mathbb{R}^{m\times n}$ and $\mathbf{u}, \mathbf{f}, \boldsymbol{\eta} \in \mathbb{R}^m$. The structure of the matrix A is crucial to define fast deconvolution algorithms since n and m are very large. In the very last years a lot of interest has been devoted to the definition of algorithms that combine edge preserving strategies with an appropriate treatment of the boundary artifacts [7,28,4,23]. In the literature one finds mainly three strategies in order to obtain both accurate and fast restorations:

(1) Choose and then impose appropriate *boundary conditions* (*BCs*) so that $n = m$ and the matrix A can be usually diagonalized by fast trigonometric transforms: discrete Fourier transform in the case of periodic BCs [21], discrete cosine transform for reflective BCs and when the blur is symmetric in any direction (quadrantally symmetric) [24,21], a low rank correction of the sine transform for anti-reflective BCs can be exploited [27,6];
(2) *Enlarge the domain* and use periodic BCs on the larger domain, such that the computations can be carried out by FFTs and eventually the image is projected back to the original domain [26,15,28];
(3) Work with the *underdetermined linear system* so that $m < n$: in such a setting the matrix A can be represented as $A = MB$, where $M \in \mathbb{R}^{m\times n}$ is a mask that selects only the valid rows of $B \in \mathbb{R}^{n\times n}$ which can be diagonalized by FFT [8,30,4,23].