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Some invariant biorthogonal sets with an application to coherent states

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ABSTRACT

We show how to construct, out of a certain basis invariant under the action of one or more unitary operators, a second biorthogonal set with similar properties. In particular, we discuss conditions for this new set to be also a basis of the Hilbert space, and we apply the procedure to coherent states. We conclude the paper considering a simple application of our construction to pseudo-Hermitian quantum mechanics.

1. Introduction

In the mathematical and physical literature many examples of complete sets of vectors in a given Hilbert space $\mathcal{H}$ are constructed starting from a single normalized element $\varphi_0 \in \mathcal{H}$, acting on this vector several time with a given set of unitary operators. For instance, this is exactly what happens for coherent states and for wavelets. In the first case one essentially acts several times on the vacuum of a bosonic oscillator with a modulation and with a translation operator. In the second example, to produce a complete set of wavelets, one acts on a *mother wavelet* with powers of a dilation and of a translation operator. In this last situation the result of this action can be an orthonormal (o.n.) set of vectors, and this is, in fact, the main output of the so-called multi-resolution analysis, [10]. On the other hand, this is forbidden for general reasons for coherent states. In two previous papers, [6,5], we have considered the following problem: given a fixed element of $\mathcal{H}$, φ_0, and a certain set of unitary operators, $A_1, \ldots, A_N$, and defining new vectors $\varphi_{k_1,\ldots,k_N} := A_1^{k_1} \cdots A_N^{k_N} \varphi_0$, $k_j \in \mathbb{Z}$ for all $j = 1, 2, \ldots, n$, is it possible to produce, out of these vectors, a new vector $\hat{\varphi}_0$ such that the new vectors $\hat{\varphi}_{k_1,\ldots,k_N} := A_1^{k_1} \cdots A_N^{k_N} \hat{\varphi}_0$ turn out to be mutually orthogonal? The answer was, in general, positive, and we have proposed an invariant procedure which, however, must be solved perturbatively. It should be mentioned that, for coherent states, that approach didn't work in all of $\mathcal{L}^2(\mathbb{R})$, but only in some suitable Hilbert subspaces of $\mathcal{L}^2(\mathbb{R})$.

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Here we consider a slightly different problem, which is also physically motivated by the recent interest on pseudo-Hermitian quantum mechanics and by the role that biorthogonal sets necessarily have in this context, because of the absence of a self-adjoint Hamiltonian describing the dynamics of the system under consideration, [7,17]. More explicitly, the problem we address in this paper is the following: is it possible, out of the linearly independent vectors $\varphi_{k_1,\ldots,k_N}$ above, to construct a new vector $\Psi_0 \in \mathcal{H}$ such that the vectors $\Psi_{k_1,\ldots,k_N} := A_1^{k_1} \cdots A_N^{k_N} \Psi_0$ are biorthogonal to the original ones? And, do these vectors define a basis in $\mathcal{H}$, at least under suitable conditions? This is not a trivial question. In fact, it is known that two biorthogonal sets are not necessarily bases, [13].

The paper is organized as follows: in the next section we state the general problem, discuss the method and show some *prototype* examples, in $N = 1$. In Section 3 we discuss in many details the case of the coherent states ($N = 2$, assuming that $A_1A_2 = A_2A_1$), and we find conditions for our procedure to work. In Section 4 we briefly discuss how to extend our procedure to $N \geqslant 3$, assuming again that the operators A_j mutually commute. Also, we consider some relations between our construction and pseudo-Hermitian quantum mechanics. Our final considerations are given in Section 5.

2. Stating the problem and first results

Let $\mathcal{H}$ be a Hilbert space, $\varphi \in \mathcal{H}$ a fixed element of the space and let $A_1, \ldots, A_N$ be N given unitary operators: $A_j^{-1} = A_j^\dagger$, $j = 1, 2, \ldots, N$. Let $\mathcal{H}_N$ be the closure of the linear span of the set

$$\mathcal{F}_\varphi = \left\{\varphi_{k_1,\ldots,k_N} := A_1^{k_1} \cdots A_N^{k_N} \varphi,\ k_1, \ldots, k_N \in \mathbb{Z}\right\}. \tag{2.1}$$

Of course, in order for this situation to be of some interest, we assume that an infinite elements of $\mathcal{F}_\varphi$ are linearly independent, so to have $\dim(\mathcal{H}_N) = \infty$. To simplify the treatment, in the following, we will assume that all the vectors $\varphi_{k_1,\ldots,k_N}$ are independent. In this case, by construction, $\mathcal{F}_\varphi$ is a basis for $\mathcal{H}_N$. However, in general, there is no reason why the vectors in $\mathcal{F}_\varphi$ should be mutually orthogonal. On the contrary, without a rather clever choice of both φ and $A_1, \ldots, A_N$, it is very unlikely to obtain an o.n. set. As stated in the introduction, our aim is to discuss some general technique which produces, for suitable A_j's, another vector $\Psi \in \mathcal{H}_N$ such that the set

$$\mathcal{F}_\Psi = \left\{\Psi_{k_1,\ldots,k_N} := A_1^{k_1} \cdots A_N^{k_N} \Psi,\ k_1, \ldots, k_N \in \mathbb{Z}\right\} \tag{2.2}$$

is biorthogonal to $\mathcal{F}_\varphi$, i.e. $\langle \varphi_{k_1,\ldots,k_N}, \Psi_{l_1,\ldots,l_N} \rangle = \delta_{k_1,l_1} \cdots \delta_{k_N,l_N}$. Moreover, we would like this set to share as much of the original features of $\mathcal{F}_\varphi$ as possible. For instance, if the set $\mathcal{F}_\varphi$ is a set of coherent states, we would like the new vectors $\Psi_{k_1,\ldots,k_N}$ to be also coherent states, in some sense, other than being stable under the action of certain unitary operators. But, first of all, we need $\mathcal{F}_\Psi$ to be a basis for $\mathcal{H}_N$.

We will consider our problem step by step, starting with the simplest situation which is, clearly, $N = 1$. In this case the set $\mathcal{F}_\varphi$ in (2.1) reduces to $\mathcal{F}_\varphi = \{\varphi_k := A^k \varphi, k \in \mathbb{Z}\}$ with $\langle \varphi_k, \varphi_l \rangle \neq \delta_{k,l}$ (otherwise we can easily solve the problem by taking $\Psi = \varphi$). Since $\mathcal{F}_\varphi$ is a basis for $\mathcal{H}_1$, any element in $\mathcal{H}_1$ can be written in terms of the vectors of $\mathcal{F}_\varphi$. Let $\Psi \in \mathcal{H}_1$ be the following linear combination:

$$\Psi = \sum_{k \in \mathbb{Z}} c_k \varphi_k, \tag{2.3}$$

and let us define more vectors of $\mathcal{H}_1$ as

$$\Psi_n := A^n \Psi = \sum_{k \in \mathbb{Z}} c_k \varphi_{k+n} = X \varphi_n, \tag{2.4}$$