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On Markov operators preserving polynomials

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A B S T R A C T

The paper is concerned with a special class of positive linear operators acting on the space $C(K)$ of all continuous functions defined on a convex compact subset K of $\mathbf{R}^d$, $d \geqslant 1$, having non-empty interior. Actually, this class consists of all positive linear operators T on $C(K)$ which leave invariant the polynomials of degree at most 1 and which, in addition, map polynomials into polynomials of the same degree. Among other things, we discuss the existence of such operators in the special case where K is strictly convex by also characterizing them within the class of positive projections. In particular we show that such operators exist if and only if ∂K is an ellipsoid. Furthermore, a characterization of balls of $\mathbf{R}^d$ in terms of a special class of them is furnished. Additional results and illustrative examples are presented as well.
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0. Introduction

The paper is concerned with a special class of positive linear operators acting on the space $C(K)$ of all continuous functions defined on a convex compact subset K of $\mathbf{R}^d$, $d \geqslant 1$, having non-empty interior. Actually, this class consists of all positive linear operators T on $C(K)$ which leave invariant the polynomials of degree at most 1 and which, in addition, map polynomials into polynomials of the same degree.

The interest for such operators comes from the study of a special differential operator $(W_T, C^2(K))$ which we carefully investigated in [6] and which is defined as

$$W_T(u) := \frac{1}{2}\sum_{i,j=1}^{d} \alpha_{ij}\frac{\partial^2 u}{\partial x_i\,\partial x_j}$$

$(u \in C^2(K))$, where $\alpha_{ij} := T(pr_i pr_j) - pr_i pr_j$ $(i, j = 1, \ldots, d)$ and each pr_i denotes the i-th coordinate function on K.

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The differential operator W_T is elliptic and it degenerates on a subset of K which contains the set of the extreme points $\partial_e K$ of K. In [6] we showed that, if T maps polynomials into polynomials of the same degree, then $(W_T, C^2(K))$ is closable in $C(K)$ and its closure generates a Markov semigroup on $C(K)$ which can be represented as a limit of suitable iterates of particular positive linear operators associated with T, namely the Bernstein–Schnabl operators associated with T, which have been deeply investigated in [5] and, more recently, in [6] and in the forthcoming monograph [7].

The main aim of the paper is to look more closely at this preservation property which seems to have an independent own interest. Among other things, we discuss the existence of such operators in the special case where K is strictly convex, i.e., $\partial_e K = \partial K$, by also characterizing them within the class of positive projections on $C(K)$ (for the bi-dimensional case see [11]). In particular we show that such operators exist if and only if ∂K is an ellipsoid. Furthermore, a characterization of balls of $\mathbf{R}^d$ in terms of a special class of them is furnished. Illustrative examples and additional results involving the tensor products and the convex convolution products of positive linear operators are presented as well.

1. Preliminaries on positive linear operators

Throughout this paper K will be a convex compact subset of $\mathbf{R}^d$, $d \geqslant 1$, with non-empty interior $\mathrm{int}(K)$. As usual we denote by $C(K)$ the space of all real-valued continuous functions on K and by $C^2(K)$ the space of all real-valued continuous functions on K that are twice continuously differentiable on $\mathrm{int}(K)$ and whose partial derivatives up to the order two can be continuously extended to K. For $u \in C^2(K)$ and $i, j = 1, \ldots, d$, we shall continue to denote by $\frac{\partial u}{\partial x_i}$ and $\frac{\partial^2 u}{\partial x_i \partial x_j}$ the continuous extensions to K of the partial derivatives $\frac{\partial u}{\partial x_i}$ and $\frac{\partial^2 u}{\partial x_i\, \partial x_j}$. The space $C(K)$, endowed with the supremum norm $\|f\|_\infty := \sup_{x\in K} |f(x)|$ $(f \in C(K))$ and the natural (pointwise) ordering, is a Banach lattice.

We also denote by $\mathbf{1}$ the constant function of constant value 1 on K and, for every $i \in \{1, \ldots, d\}$, by pr_i the i-th coordinate function on K, i.e., $pr_i(x) = x_i$ for every $x = (x_i)_{1\leqslant i\leqslant d} \in K$.

Let B_K be the σ-algebra of all Borel subsets of K and denote by $M^+(K)$ (resp., $M_1^+(K)$) the subset of all Borel measures (resp., the subset of all probability Borel measures) on K. In particular, for every $x \in K$, the symbol ε_x stands for the unit mass concentrated at x, i.e., for every $B \in B_K$,

$$\varepsilon_x(B) := \begin{cases} 1 & \text{if } x \in B; \\ 0 & \text{if } x \notin B. \end{cases}$$

If $\tilde{\mu} \in M^+(K)$, then $Supp(\tilde{\mu})$ denotes the support of $\tilde{\mu}$, i.e., the complement of the largest open subset of K having measure zero with respect to $\tilde{\mu}$.

Given a *Markov operator* $T : C(K) \to C(K)$, i.e., a positive linear operator such that $T(\mathbf{1}) = \mathbf{1}$, by the Riesz representation theorem there exists a unique family $(\tilde{\mu}_x^T)_{x\in K}$ in $M_1^+(K)$ such that

$$T(f)(x) = \int_K f\, d\tilde{\mu}_x^T \quad \bigl(f \in C(K),\ x \in K\bigr). \tag{1.1}$$

Such a family is said to be the *continuous selection of probability Borel measures on K associated with T*. By means of $(\tilde{\mu}_x^T)_{x\in K}$ we can construct the so-called *Bernstein–Schnabl operators associated with T* which are defined by setting, for every $n \geqslant 1$, $x \in K$ and $f \in C(K)$,

$$B_n(f)(x) = \int_K \cdots \int_K f\left(\frac{x_1 + \cdots + x_n}{n}\right) d\tilde{\mu}_x^T(x_1) \cdots d\tilde{\mu}_x^T(x_n). \tag{1.2}$$

Note that by the continuity property of the product measure it follows that $B_n(f) \in C(K)$. Moreover, $B_1 = T$.

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