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On the rotational invariance for the essential spectrum of λ-Toeplitz operators

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ABSTRACT

Let λ be a complex number in the closed unit disc, and $\mathcal{H}$ be a separable Hilbert space with the orthonormal basis, say, $\mathcal{E} = \{e_n\colon\ n = 0, 1, 2, \ldots\}$. A bounded operator T on $\mathcal{H}$ is called a *λ-Toeplitz operator* if $\langle Te_{m+1}, e_{n+1}\rangle = \lambda\langle Te_m, e_n\rangle$ (where $\langle\cdot,\cdot\rangle$ is the inner product on $\mathcal{H}$). The L^2 function $\varphi \sim \sum a_n e^{in\theta}$ with $a_n = \langle Te_0, e_n\rangle$ for $n \geqslant 0$ and $a_n = \langle Te_n, e_0\rangle$ for $n < 0$ is, on the other hand, called the *symbol of* T. Let us denote T by $T_{\lambda,\varphi}$. It can be verified directly that $T_{\lambda,\varphi}$ is an "eigenoperator" associated with the eigenvalue λ for the following map on $\mathcal{B}(\mathcal{H})$:

$$\phi(A) = S^*AS, \quad A \in \mathcal{B}(\mathcal{H}),$$

where S is the unilateral shift defined by $Se_n = e_{n+1}$, $n = 0, 1, 2, \ldots$. In an earlier joint work, the author used a result of M.T. Jury regarding the Fredholm theory of a certain Toeplitz-composition C^*-algebra to show that if φ is in the class C^1 and if $|\lambda| = 1$ has finite order, then the essential spectrum of $T_{\lambda,\varphi}$ is "rotationally invariant" with respect to λ, i.e.,

$$\sigma_e(T_{\lambda,\varphi}) = \lambda\sigma_e(T_{\lambda,\varphi}).$$

In this paper, we prove that the C^1 restriction for the symbol φ in the above result can be dropped entirely, and the equation actually holds for any φ in L^∞ and any $|\lambda| = 1$. It turns out that the key for removing the assumption on the smoothness of φ depends only on the definition of $T_{\lambda,\varphi}$ and some very elementary properties of S as a Fredholm operator. The applications of this phenomenon for $\sigma_e(T_{\lambda,\varphi})$ include a generalization of A. Wintner's result on the spectra of Toeplitz operators with bounded analytic symbols.

1. Introduction

Let $\mathcal{H}$ be a separable Hilbert space with an orthonormal basis, say, $\mathcal{E} = \{e_n\colon\ n = 0, 1, 2, \ldots\}$. Given $\lambda \in \overline{\mathbb{D}} = \{z \in \mathbb{C}\colon |z| \leqslant 1\}$, a bounded operator T is called a *λ-Toeplitz operator* if $\langle Te_{m+1}, e_{n+1}\rangle = \lambda\langle Te_m, e_n\rangle$ (where $\langle\cdot,\cdot\rangle$ is the inner product on $\mathcal{H}$). In terms of the basis $\mathcal{E}$, it is easy to see that the matrix representation of T is given by

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$$\begin{pmatrix} a_0 & a_{-1} & a_{-2} & a_{-3} & a_{-4} & \cdots \\ a_1 & \lambda a_0 & \lambda a_{-1} & \lambda a_{-2} & \lambda a_{-3} & \ddots \\ a_2 & \lambda a_1 & \lambda^2 a_0 & \lambda^2 a_{-1} & \lambda^2 a_{-2} & \ddots \\ a_3 & \lambda a_2 & \lambda^2 a_1 & \lambda^3 a_0 & \lambda^3 a_{-1} & \ddots \\ a_4 & \lambda a_3 & \lambda^2 a_2 & \lambda^3 a_1 & \lambda^4 a_0 & \ddots \\ \vdots & \ddots & \ddots & \ddots & \ddots & \ddots \end{pmatrix}$$

for some double sequence $\{a_n\colon n \in \mathbb{Z}\}$, and the boundedness of T clearly implies that $\sum |a_n|^2 < \infty$. Therefore, it is natural to introduce the notation

$$T = T_{\lambda,\varphi},$$

where $\varphi \sim \sum_{-\infty}^{\infty} a_n e^{in\theta}$ (the *symbol* of $T_{\lambda,\varphi}$) belongs to $L^2 = L^2(\mathbb{T})$, the Hilbert space of square integrable functions with respect to the Lebesgue measure on the unit circle $\mathbb{T}$ whose inner product is defined by

$$\langle f, g\rangle = \frac{1}{2\pi}\int_0^{2\pi} f\bar{g}\, d\theta,$$

and consider $T_{\lambda,\varphi}$ as an operator acting on the *Hardy space*

$$H^2 = \left\{ f \in L^2\colon \int_0^{2\pi} f\left(e^{i\theta}\right) e^{in\theta}\, d\theta = 0,\ n < 0 \right\},$$

with the identification $\mathcal{H} = H^2$ and e_n identified with the function $e^{in\theta}$, $n \geqslant 0$.

It can be checked easily that if $f \sim \sum_0^{\infty} a_n e^{i\theta} \in H^2$, then the analytic function

$$\hat{f}(z) := \frac{1}{2\pi}\int_0^{2\pi} \frac{f(e^{i\theta})}{1 - ze^{-i\theta}}\, d\theta, \quad |z| < 1,$$

equals the analytic function defined by the series $\sum_0^{\infty} a_n z^n$, $|z| < 1$, and, by classical Fourier analysis and a theorem of Fatou, $\hat{f}_r(e^{i\theta}) = \hat{f}(re^{i\theta}) \to f(e^{i\theta})$ a.e. θ and in L^2 as $r \to 1$. Hence H^2 is often identified with the Hilbert space of analytic functions $\{\hat{f}\colon\ f \in H^2\}$, with inner product

$$\langle \hat{f}, \hat{g}\rangle = \frac{1}{2\pi}\int_0^{2\pi} f\bar{g}\, d\theta,$$

and therefore the notations $f(e^{i\theta})$ or $f(z)$ will be used freely in this article, and we sometimes call $f(e^{i\theta})$ the *boundary value function* of the analytic function $f(z)$ (for the theory of Hardy spaces, we refer the readers to Chapter 6 of [4] and [8]).

Other known spaces of functions considered in this paper include: (1) $L^\infty = L^\infty(\mathbb{T})$, the space of bounded Lebesgue measurable functions on $\mathbb{T}$; (2) $C(\mathbb{T})$, the space of continuous functions on $\mathbb{T}$; (3) $C^1(\mathbb{T})$, the space of continuously differentiable functions on $\mathbb{T}$; (4) $H^\infty = H^\infty(\mathbb{D})$, the space of bounded analytic functions in $\mathbb{D}$; (5) $\mathbb{A} = \mathbb{A}(\mathbb{D})$, the algebra of continuous functions on $\overline{\mathbb{D}}$ which are analytic in $\mathbb{D}$. Moreover, $\mathbb{A}$ and H^∞ can be regarded as linear subspaces of H^2, and we have $\mathbb{A} \subsetneq H^\infty \subsetneq H^2$.

Also note that when $\lambda = 1$ and $\varphi \in L^\infty$, the matrix of $T_{1,\varphi}$ is the matrix of the bounded Toeplitz operator T_φ on H^2. For the readers who are not familiar with the operator theory on H^2, the *Toeplitz operator T_φ with symbol $\varphi \in L^\infty$* is the operator defined by $T_\varphi f = P(\varphi f)$, $f \in H^2$, where P is the projection from L^2 on to H^2. Here we refer the reader to Chapter 7 of [4] for the theory of Toeplitz operators.

λ-Toeplitz operators arise in a natural way as the "eigenoperators" for the following map on $\mathcal{B}(\mathcal{H})$ (space of bounded operators on $\mathcal{H}$):

$$\phi(A) = S^*AS, \quad A \in \mathcal{B}(\mathcal{H}),$$

where S is the unilateral shift on $\mathcal{H}$, i.e., $Se_n = e_{n+1}$, $n = 0, 1, 2, \ldots,$ as the equation $S^*T_{\lambda,\varphi}S = \lambda T_{\lambda,\varphi}$ can be verified directly. In fact, the map ϕ was considered at the end of the paper [1] by Barría and Halmos as a possible topic for future study.