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Avoiding σ-porous sets in Hilbert spaces

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ABSTRACT

We give a constructive proof that any σ-porous subset of a Hilbert space has Lebesgue measure zero on typical C^1 curves. Further, we discover that this result does not extend to all forms of porosity; we find that even power-p porous sets may meet many C^1 curves in positive measure.

1. Introduction

Porous and σ-porous sets form a class of exceptional sets which arise naturally in the study of differentiability of convex and Lipschitz functions. The concept of σ-porous sets was introduced by Dolženko in [2], and the connection between porosity and differentiability is investigated in [4,7,8]. We introduce, in the following definition, various types of porosity which we will work with in this paper.

Definition 1.1. Let (M, d) be a metric space, $c \in (0, 1)$ and $p > 1$.

(i) A subset E of M is said to be c-porous at a point $x \in E$ if for every $\epsilon > 0$ there exists a point $h \in M$ and a real number $r > 0$ such that

$$d(x, h) < \epsilon, \qquad B(h, r) \cap E = \emptyset \quad \text{and} \quad r > c \cdot d(x, h).$$

Here $B(h, r)$ denotes the open ball in M with centre h and radius r.

(ii) We say that E is a c-porous subset of M if E is c-porous at every point $x \in E$. A subset E of M is called porous if it is c-porous for some $c \in (0, 1)$.

(iii) A subset P of M is said to be σ-porous if it can be expressed as a countable union of porous subsets of M.

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(iv) A subset P of M is said to be power-p-porous at a point $x \in P$ if whenever $\epsilon > 0$, there exists $h \in M$ and $r > 0$ such that $d(h, x) < \epsilon$, $B(h, r) \cap P = \emptyset$ and $r > d(h, x)^p$. P is called power-p-porous if P is power-p-porous at every point $x \in P$.

An immediate consequence of the definition is that σ-porous sets are of first category. Moreover, using the Lebesgue density theorem, σ-porous sets in finite dimensional spaces have Lebesgue measure zero. However, in infinite dimensional settings, where there is no analogue of the Lebesgue measure (see [3]), it is more difficult to describe the size of σ-porous sets. In our main result we exhibit a phenomenon of the σ-porous subsets of Hilbert spaces which reveals that they are also, in some sense, very small. Namely that any σ-porous subset of a Hilbert space is avoided by typical curves. More information about porosity and σ-porosity can be found in the survey [11], and, for a broader introduction to negligible sets relating to differentiability, see [1, Chapter 6].

The notion of a set being Γ_n null was introduced by Lindenstrauss, Preiss and Tišer in [4, Chapter 5]. For a Banach space X, we denote by $\Gamma_n(X)$ the Banach space $C^1([0,1]^n, X)$ of continuously differentiable maps from $[0,1]^n$ to X.

Definition 1.2. A Borel subset A of a Banach space X is called Γ_n-null if

$$\mathcal{L}^n\big\{t \in [0,1]^n\colon\ \gamma(t) \in A\big\} = 0$$

for residually many $\gamma \in \Gamma_n(X)$.

It has been established, in the recent book [4] of Lindenstrauss, Preiss and Tišer, that every σ-porous subset of a separable Banach space having a separable dual is Γ_1-null [4, Theorem 10.4.1]. In the present paper we give a new proof for σ-porous sets in Hilbert spaces. Our argument is noteworthy because it is constructive and presents a method of finding many curves which avoid a given porous set.

There have been notable discoveries of the opposite nature. In, [9] Speight proves that for every Banach space X and integer n with $2 < n < \dim X$, there exists a directionally porous set $P \subset X$ which is not Γ_n-null. Thus, our main result fails if we try to replace curves with n-dimensional surfaces, where $n > 2$. Further, [5] verifies the existence of an 'unavoidable' σ-porous set – a σ-porous set whose complement is null on all Lipschitz curves. Such a set can be found in any Banach space containing l_1.

1.1. Structure of the paper

We work in a fixed Hilbert space H. In Section 2, we introduce a method of altering a given curve in $\Gamma_1(H)$ in order to obtain a nearby curve which avoids a given porous set E. This is achieved by pushing segments of the original curve inside the holes of E. Critical to our approach will be an application of the Vitali Covering Theorem [6, Chapter 2] which allows us to nominate suitable segments of the curve to modify.

The crucial lemma (Lemma 3.1) in the proof of our main result is stated at the beginning of Section 3 and the majority of this section is devoted to its proof. We present an algorithm in which we repeatedly apply the results established in Section 2, in order to construct a finite sequence of curves so that the final curve is close to the original and we have good control over the measure of a porous set E on all curves in some neighbourhood. The biggest problem we face is ensuring that the difference between the final curve and our original does not become too large. We overcome this difficulty by ensuring that the difference in the derivatives behaves like a martingale. This allows us to control the size of the set where the difference in derivatives becomes too large. Here, we require Kolmogorov's Martingale Theorem, quoted below from [10, p. 237]: