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Linear combinations of frame generators in systems of translates

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ABSTRACT

A finitely generated shift invariant space V is a closed subspace of $L^2(\mathbb{R}^d)$ that can be generated by the integer translates of a finite number of functions. A set of frame generators for V is a set of functions whose integer translates form a frame for V. In this note we give necessary and sufficient conditions in order that a minimal set of frame generators can be obtained by taking linear combinations of the given frame generators. Surprisingly the results are very different from the recently studied case when the property to be a frame is not required.

1. Introduction

Shift invariant spaces (SISs) are closed subspaces of $L^2(\mathbb{R}^d)$ that are invariant under integer translations. They play an important role in approximation theory, harmonic analysis, wavelet theory, sampling and signal processing [1,9,12,15]. The structure of these spaces has been deeply analyzed (see for example [6,7,3,11,16]).

A set of functions Φ is a *set of generators* for a shift invariant space V if the closure of the space spanned by all integer translations of the functions in Φ agrees with V. When there exists a finite set of generators Φ for V, we say that V is finitely generated. In this case, there exists a positive integer ℓ, called the length of V, that is defined as the minimal number of functions that generate V. Any set of generators of V with ℓ elements will be called a *minimal set of generators*.

Let $\Phi = \{\phi_1, \dots, \phi_m\}$ be a set of generators for a shift invariant space V. It is interesting to know whether it is possible to obtain a minimal set of generators from the given generators in Φ. There are many examples

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with the property that no subset of Φ is a minimal set of generators. So, deleting elements from Φ may not be a successful procedure.

Concerning this question, Bownik and Kaiblinger in [4], showed that a minimal set of generators for V can be obtained from Φ by linear combinations of its elements. Moreover, they proved that almost every set of ℓ functions that are linear combinations of $\{\phi_1,\dots,\phi_m\}$ is a minimal set of generators for V (see [4, Theorem 1]). We emphasize that the linear combinations only involve the functions $\{\phi_1,\dots,\phi_m\}$ and not their translations.

Since linear combinations of a finite number of functions preserve properties such as smoothness, compact support, bandlimitedness, decay, etc., an interesting consequence of Bownik and Kaiblinger's result is that if the generators for V have some additional property, there exists a minimal set of generators that inherits this property.

In many problems involving shift invariant spaces, it is important that the system of translates $\{T_k\phi_j\colon k\in\mathbb{Z}^d,\ j=1,\dots,m\}$ bears a particular functional analytic structure such as being an orthonormal basis, a Riesz basis or a frame. Therefore, it is interesting to know when a minimal set of generators obtained by taking linear combinations of the original one has the same structure. More precisely, suppose that $\Phi=\{\phi_1,\dots,\phi_m\}$ generates a shift invariant space V of length ℓ, and assume that a new set of generators $\Psi=\{\psi_1,\dots,\psi_\ell\}$ for V is produced by taking linear combinations of the functions in Φ. That is, assume that $\psi_i=\sum_{j=1}^m a_{ij}\phi_j$ for $i=1,\dots,\ell$ for some complex scalars a_{ij}. If we collect the coefficients in a matrix $A=\{a_{ij}\}_{i,j}\in\mathbb{C}^{\ell\times m}$, then we can write in matrix notation $\Psi=A\Phi$. We would like to know, which matrices A transfer the structure of Φ over Ψ. Precisely, we study the following question: If we know that $\{T_k\phi_i\colon k\in\mathbb{Z}^d,\ i=1,\dots,m\}$ is a frame for V, when will $\{T_k\psi_i\colon k\in\mathbb{Z}^d,\ i=1,\dots,\ell\}$ also be a frame for V?

In this paper we answer this question completely. As we mentioned before, the property of being a "set of generators" for a SIS V is generically preserved by the action of a matrix A [4]. This is not anymore valid for the case of frames. This is an unexpected result. More than that, we were able to construct a surprising example of a shift invariant space V with a set of generators Φ such that their integer translates $\{T_k\phi_i\colon k\in\mathbb{Z}^d,\ i=1,\dots,m\}$ form a frame for V and with the property that no matrix A, of size $\ell\times m$ with $\ell<m$, transform Φ into a new set of generators that form a frame for V.

Our main result gives exact conditions in order that the frame property is preserved by a matrix A. These conditions are in terms of a particular geometrical relation that has to be satisfied between the nullspace of A and the column space of $G_\Phi(\omega)$ for almost all ω. The proof uses recent results about singular values of composition of operators and involves the Friedrichs angle between subspaces in Hilbert spaces. We also provide an equivalent analytic condition between A and $G_\Phi(\omega)$ in order that this same result holds. Although we are interested in the case $\ell=\ell(V)$ (the length of the SIS V under study), most of our results are still valid when $\ell(V)\leqslant\ell\leqslant m$. For completeness we include the particular case of Riesz bases and orthonormal bases that are known.

The paper is organized as follows. In Section 2 we set the definitions and results that we need. We include some results about the eigenvalues of conjugated matrices in Section 3. Finally, in Section 4 we state and prove our main results.

2. Preliminaries

We start this section by giving the basic definitions. Then, we state some known result about shift invariant spaces that we will need later.

Definition 2.1. Let $\mathcal{H}$ be a separable Hilbert space and $\{f_k\}_{k\in\mathbb{Z}}$ be a sequence in $\mathcal{H}$.

(a) The sequence $\{f_k\}_{k\in\mathbb{Z}}$ is said to be a *Riesz basis* for $\mathcal{H}$ if it is complete in $\mathcal{H}$ and if there exist $0<\alpha\leqslant\beta$ such that for every finite scalar sequence $\{c_k\}_{k\in\mathbb{Z}}$ one has