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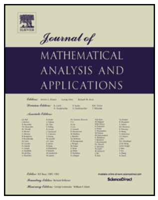

The global Cauchy problems for the nonlinear dispersive equations on modulation spaces

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ABSTRACT

In this paper, we discuss decay estimates and Strichartz estimates for dispersive equations with non-homogeneous symbols on modulation spaces $M^s_{p,q}$ to obtain the global well-posedness of the Cauchy problems for nonlinear dispersive equations. As a result, we have a generalization of the result in [19] which treated the Schrödinger equations with a nonlinearity of wider class.

1. Introduction

In this paper, we study the Cauchy problems for nonlinear dispersive equations:

$$\begin{cases} i\partial_t u + \phi(\sqrt{-\Delta}\,)u = f(u), \\ u(0) = u_0, \end{cases} \tag{1}$$

where $u(x,t) : \mathbf{R}^n \times \mathbf{R} \to \mathbf{C}$, $u_0(x) : \mathbf{R}^n \to \mathbf{C}$, $f(u) : \mathbf{C} \to \mathbf{C}$ is a nonlinear function, $i = \sqrt{-1}$, $\Delta = \partial^2/\partial^2_{x_1} + \cdots + \partial^2/\partial^2_{x_n}$, $n \geqslant 1$, and $\phi(\sqrt{-\Delta}\,) = \mathcal{F}^{-1}\phi(|\xi|)\mathcal{F}$ is a Fourier multiplier. The Cauchy problem (1) can express many kinds of dispersive equations. For example, $\phi(r) = r$, $\phi(r) = r^2$, $\phi(r) = \sqrt{1+r^2}$, and $\phi(r) = \sqrt{1+r^4}$ correspond to the wave equation, the Schrödinger equation, the Klein–Gordon equation, and the beam equation, respectively. By Duhamel's principle, the solutions of the Cauchy problem (1) satisfy

$$u(t) = U(t)u_0 - i\int_0^t U(t-s)f(u)\,ds$$

where

$$U(t) = e^{it\phi(\sqrt{-\Delta}\,)}.$$

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So, if we get the solutions of these integral equations, then we have the solutions of the Cauchy problem (1).

In order to solve the Cauchy problems for nonlinear dispersive equations, decay estimates and the Strichartz estimates play an important role. As the most simple and important example of the Cauchy problem (1), we consider the Cauchy problem for the Schrödinger equation. Recall that the Schrödinger semi-group $e^{it\Delta}$ on Lebesgue spaces L^p has the following decay estimates:

$$\left\|e^{it\Delta}f\right\|_p \lesssim |t|^{-n(1/2-1/p)}\|f\|_{p'}, \quad 2 \leqslant p \leqslant \infty. \tag{2}$$

From these estimates, we obtain the Strichartz estimates:

$$\left\|e^{it\Delta}f\right\|_{L^\gamma(\mathbf{R},L^p)} \lesssim \|f\|_2, \quad 2 \leqslant p \leqslant \infty,$$

where $2/\gamma = n(1/2 - 1/p)$ and $\|f\|_{L^\gamma(\mathbf{R},L^p)} = (\int_{\mathbf{R}} \|f(t)\|_p^\gamma\, dt)^{1/\gamma}$. By a fixed point theorem and these two estimates, we can show the well-posedness of nonlinear Schrödinger equations (cf. [2,3,1,8]). There are two kinds of the well-posedness: the one obtained from the decay estimates and the one from the Strichartz estimates. For example, Cazenave and Weissler [4] showed by the decay estimates that there exists a solution u of the nonlinear Schrödinger equation with $f(u) = |u|^\kappa u$ and sufficiently small u_0; $\sup_{t\in\mathbf{R}} |t|^B \|e^{it\Delta}u_0\|_{L^{2+\kappa}} < +\infty$ such that

$$\|u\|_X = \sup_{t\in\mathbf{R}} |t|^B \left\|u(t)\right\|_{L^{2+\kappa}} < +\infty,$$

where $\kappa \in \mathbf{R}$, $\kappa_0 < \kappa < 4/(n-2)$ ($\kappa_0 < \kappa < \infty$ if $n = 1$), $B = \frac{4-(n-2)\kappa}{2\kappa(\kappa+2)}$, $0 < B < n\kappa/2(2+\kappa)$, and $B(1+\kappa) < 1$. κ_0 is the positive root of the equation: $n\kappa_0^2 + (n-2)\kappa_0 - 4 = 0$. On the other hand, using the Strichartz estimates, Cazenave and Weissler [2] studied the local L^2 critical case, i.e. $\kappa = 4/n$, and the local H_2^1 critical case, i.e. $\kappa = 4/(n-2)$, for the Schrödinger equation. Moreover, in [3], they discussed the local H_2^s critical case, i.e. $\kappa = 4/(n-2s)$, and obtained the global solution for sufficiently small initial data in $\dot{H}_2^s$.

Wang and Hudzik [19] obtained the global well-posedness of the nonlinear Schrödinger equation on modulation spaces $M_{p,q}^s$. On modulation spaces, decay estimates of Schrödinger semi-group have the following forms:

$$\left\|e^{it\Delta}f\right\|_{M_{p,q}^s} \lesssim \left(1+|t|\right)^{-n(1/2-1/p)}\|f\|_{M_{p',q}^s}, \quad 2 \leqslant p \leqslant \infty,\ 1 \leqslant q \leqslant \infty,\ s \in \mathbf{R}. \tag{3}$$

In comparison to the decay estimates (2), estimates (3) do not have a singular point at $t = 0$. Moreover, the decay rate of the estimates (3) at $t = \infty$ is the same as that of the estimates (2). By virtue of this property, if we consider the global well-posedness on modulation spaces when $f(u)$ is a $(\kappa+1)$-time product of u and $\bar{u}$, we can show that there exists a solution u such that

$$\|u\|_X = \sup_{t\in\mathbf{R}} \langle t\rangle^{2/\gamma} \left\|u(t)\right\|_{M_{2+\kappa,1}} < +\infty$$

where $\kappa \in \mathbf{N}$, $\kappa_0 < \kappa$ and $2/\gamma = n\kappa/2(2+\kappa)$. That is, the use of modulation spaces enables us to deal with the case $\kappa \geqslant 4/(n-2)$ (cf. [19]). For precise definition of modulation spaces $M_{p,q}^s$, see Section 2. In recent works, inclusion relations between modulation and the other spaces have been studied. For example, we have easily $M_{p,1} \subset L^p \subset M_{p,\infty}$, for any $1 < p < \infty$. In addition, embedding between modulation and Besov spaces is studied by Gröbner [6], Toft [17], Sugimoto and Tomita [16]. Moreover, Sugimoto and Kobayashi [13] studied embedding between modulation and L^p-Sobolev spaces. Some other properties of Besov spaces can be found in [18].