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Powers and direct sums

John B. Conway [a,*], Gabriel Prăjitură [b], Alejandro Rodríguez-Martínez [c]

[a] The George Washington University, United States
[b] SUNY Brockport, United States
[c] Khalifa University of Science, Technology and Research, Abu Dhabi, United Arab Emirates

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ABSTRACT

This paper investigates when a bounded operator on Hilbert space has the property that its square is similar to to the direct sum of it with itself. A few general results are obtained and the normal operators and unilateral shifts having this property are characterized. Several additional examples are explored.

1. Introduction

If a general theory of multiplicity ever comes to be, it will likely be the case that for any operator A, the operator $A^{(2)} = A \oplus A$ has twice the multiplicity of A. As in the case of hermitian operators, it might sometimes be the case that A^2 has twice the multiplicity of A and sometimes that it does not. Though this paper will not try to begin a general theory of multiplicity, it does explore the relationship between $A \oplus A$ and A^2. Specifically it studies operators A such that $A \oplus A$ and A^2 are similar, a question that has an intrinsic interest independent of any attempt at multiplicity theory.

1.1. Definition. If $A \in \mathcal{B}(\mathcal{H})$, A satisfies *Condition S* if $A^2 \approx A \oplus A$. Say that A satisfies *Condition U* if $A^2 \cong A \oplus A$. ($\approx$ means similar and $\cong$ means unitarily equivalent.)

Realize that for normal operators Condition S and Condition U are the same since two normal operators are similar if and only if they are unitarily equivalent [1, Corollary IX.6.11].

We will shortly see several examples of operators satisfying Condition S, but for the time being we present only two.

1.2. Example. If S is the unilateral shift, then S satisfies Condition U.

1.3. Example. If A denotes multiplication by the independent variable on $L^2[0,1]$, then A does not satisfy Condition S. In fact since x^2 is one-to-one on $[0,1]$, A^2 is unitarily equivalent to A; however $A \oplus A$ has uniform multiplicity 2. On the other hand, $A^{(\infty)}$, the direct sum of A with itself an infinite number of times, does satisfy Condition S (and therefore Condition U).

The paper is organized as follows. Section 2 presents some spectral properties of operators that satisfy Condition S. In particular it is shown that such operators have spectral radius 1 and that the only compact operator satisfying Condition S

* Corresponding author.
E-mail addresses: conway@gwu.edu (J.B. Conway), gprajitu@brockport.edu (G. Prăjitură), Alejandro.Rodriguez@kustar.ac.ae (A. Rodríguez-Martínez).

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is the zero operator. Section 3 gives a complete characterization of the normal operators satisfying Condition S. In Section 4 the unilateral weighted shifts satisfying Condition S as well as those satisfying Condition U are characterized. This results in a rather simple criterion for a hyponormal weighted shift to satisfy Condition S and the fact that the isometric weighted shift is the only hyponormal weighted shift that satisfies Condition U.

2. Spectral results

The proofs of the next two results are straightforward. Recall that $\sigma_e(A)$ denotes the essential spectrum of A; that is, the spectrum of the image of A in the Calkin algebra.

2.1. Proposition. *If A satisfies Condition S, then $\sigma(A) = \sigma(A)^2$ and $\sigma_e(A) = \sigma_e(A)^2$.*

2.2. Proposition. (a) *If A and B satisfy Condition S (or U), then so does $A \oplus B$.*
(b) *A satisfies Condition S (or U) if and only if A^* does.*
(c) *If A satisfies Condition S (or U), then so does A^{2^n} for all $n \in \mathbb{N}$.*

Note that a scalar multiple of an operator satisfying Condition S (or U) satisfies the same condition only if the scalar is either 0 or 1.

In light of Proposition 2.1 it becomes important for the problem of characterizing the operators satisfying Condition S to characterize the compact subsets K of $\mathbb{C}$ that satisfy $K = K^2$. At present we cannot do this, but we can make such a characterization when K is a subset of $\mathbb{R}$.

2.3. Proposition. *A compact subset K of $\mathbb{R}$ satisfies $K = K^2$ if and only if one of the following holds*:
(a) $K = \{0\}$;
(b) $K = \{1\}$;
(c) $K = \{0, 1\}$;
(d) *for any r in the open interval $(0, 1)$ there is a compact subset D of $[r^2, r]$ such that $K = \bigcup_{n=0}^{n=\infty} D^{2^n} \cup \bigcup_{n=1}^{n=\infty} D^{1/2^n} \cup \{0, 1\}$.*

Proof. It is clear that any set K that has the form in parts (a) through (d) must satisfy $K = K^2$, so we look at the converse. Assume $K = K^2$ and assume that none of the conditions (a) through (c) is true; let $0 < r < 1$. We first note that $K \cap [r^2, r] \neq \emptyset$. In fact if $s \in K$ and $s \neq 0, 1$, let n be the smallest natural number such that $s^{2^n} \leqslant r$. If it were the case that $s^{2^n} \leqslant r^2$, then we would have that $s^{2^{n-1}} \leqslant r$, a contradiction; so $s^{2^n} > r^2$. That is $D = K \cap [r^2, r] \neq \emptyset$. Let $L = \bigcup_{n=0}^{n=\infty} D^{2^n} \cup \bigcup_{n=1}^{n=\infty} D^{1/2^n} \cup \{0, 1\}$. Clearly $L \subseteq K$ and $L = L^2$. Using an argument similar to the one used to show that $K \cap [r^2, r] \neq \emptyset$ we can establish the reverse inclusion and thus the equality of K and L. □

The next result is straightforward.

2.4. Proposition. *Assume E is a non-empty, bounded subset of $\mathbb{C}$ that satisfies $E = E^2$.*
(a) $E \subseteq \operatorname{cl} \mathbb{D}$.
(b) *If $E \cap \mathbb{D} \neq \emptyset$, then $0 \in \operatorname{cl} E$ and $\operatorname{cl} E \cap \partial\mathbb{D} \neq \emptyset$.*
(c) *If $\operatorname{int} E \neq \emptyset$ and $E = -E$, then $(\operatorname{int} E)^2 = \operatorname{int} E$.*
(d) $(\operatorname{cl} E)^2 = \operatorname{cl} E$.

The next result illustrates the utility of an extra assumption on sets E such that $E = E^2$. The authors wish to thank the referee for this suggestion.

2.5. Lemma. *If $E \subseteq \operatorname{cl} \mathbb{D}$ and $E = E^2 = -E$, then whenever $a \in E$, E contains a dense set of the circle $\{z\colon |z| = |a|\}$.*

Proof. When $a = |a|e^{i\alpha} \in E$, let $T_a = \{\theta \in [0, 2\pi]\colon ae^{i\theta} \in E\}$. The lemma will follow by establishing the following.

Claim. *If $\theta \in T_a$, then $\theta + \frac{k\pi}{2^n} \in T_a$ for $0 \leqslant k \leqslant 2^{n+1}$, where the addition is modulo 2π.*

This is established by induction. Assume $n = 1$. If $\theta \in T_a$, then $|a|e^{i(\alpha+\theta)} \in E$; thus $-|a|^2 e^{i(2\alpha+2\theta)} \in E$. Taking square roots it follows that $i|a|e^{i(\alpha+\theta)} = |a|e^{i(\alpha+\theta+\pi/2)} \in E$ and $|a|e^{i(\alpha+\theta+3\pi/2)} \in E$. That is, $\theta + \frac{\pi}{2}, \theta + \frac{3\pi}{2} \in T_a$. This says that $\theta + \frac{k\pi}{2} \in T_a$ for $0 \leqslant k \leqslant 4$ since the cases $k = 2, 4$ are trivial. The proof of the induction step is similar. □

Note that if E is as in the preceding lemma, then so is $\operatorname{cl} \mathbb{D} \setminus E$.