J. Math. Anal. Appl. 413 (2014) 953–962

Contents lists available at ScienceDirect

Journal of Mathematical Analysis and Applications

www.elsevier.com/locate/jmaa

Log-concavity for Bernstein-type operators using stochastic orders

F.G. Badía *, C. Sangüesa

Departamento de Métodos Estadísticos, Universidad de Zaragoza, Zaragoza, Spain

ARTICLE INFO

Article history:
Received 22 November 2012
Available online 11 December 2013
Submitted by R. Stelzer

Keywords:
Log-concavity
Bernstein-type operator
Szász operator
Gamma operator
Stochastic order
Incomplete Gamma function

ABSTRACT

This paper aims to study the preservation of log-concavity for Bernstein-type operators. In particular, attention is focused on positive linear operators, defined on the positive semi-axis, admitting a probabilistic representation in terms of a process with independent increments. This class includes the classical Gamma, Szász, and Szász–Durrmeyer operators. With respect to the first and second operators, the results of this paper correct two erroneous counterexamples in [10]. As a main tool in our results we use stochastic order techniques. Our results include, as a particular case, the log-concavity of certain functions related to the incomplete Gamma function.

1. Introduction

In this paper we consider positive linear operators of Bernstein-type, defined on the positive semi-axis, admitting a probabilistic representation in terms of Lévy processes (that is, a process with independent and stationary increments). Our aim is to prove the preservation of log-concavity for these operators, by using basic characteristics of the underlying processes, together with some results coming from the well developed theory of stochastic orders in the field of probability theory (cf. [28]).

Log-concave functions (or Pólya frequency functions of order two, see [19, Chapter 7]) are a class of functions arising in many different fields, such as economics (see, e.g., [8,12,16]), statistics (see, e.g., [14,30]), and applied probability (see, e.g., [13,27]).

Probabilistic methods have played a significant role in the analysis of the preservation of shape and approximation properties of Bernstein-type operators, especially since the last years of the last century. Illustrative examples of this methodology can be found in [1,3,9,20,21].

Following this approach, in [10] the preservation of log-convexity for the Szász, Gamma centered, Baskakov, and Weierstrass operators is shown, and that of log-concavity for the last. The methodology used in [10] cannot be applied to the preservation of log-concavity of these operators as the Cauchy–Schwartz inequality works in the reverse way. In this paper we use stochastic orders as a tool to prove the preservation of log-concavity for some Bernstein operators such as the Szász and Gamma operators. Stochastic orders, apart from its natural use in applied probability, have interest in many areas of mathematics. The papers [4,5,15] are examples of the use of stochastic orders in the analysis of Bernstein-type operators.

The two first classical operators considered (Szász and Gamma-type) are defined in terms of the Poisson and Gamma kernel, given respectively by

* Corresponding author.
E-mail addresses: gbadia@unizar.es (F.G. Badía), csangues@unizar.es (C. Sangüesa).

0022-247X/$ – see front matter
http://dx.doi.org/10.1016/j.jmaa.2013.12.014

$$p_t(k) = e^{-t}\frac{t^k}{k!}, \quad k = 0, 1, \ldots, \quad \text{and} \quad g_t(u) = \frac{u^{t-1}}{\Gamma(t)}e^{-u}, \quad u > 0,\ t > 0, \tag{1}$$

in which $\Gamma(\cdot)$ denotes the Gamma function.

In this way, the Szász and Gamma operators are defined, for each $t > 0$, by

$$Lf(t) = \sum_{k=0}^{\infty} f(k)p_t(k), \quad \text{Szász operator}, \tag{2}$$

$$Gf(t) = \int_0^{\infty} f(u)g_t(u)\,du, \quad \text{Gamma operator}, \tag{3}$$

where f is a suitable function defined on the positive real semi-axis.

Note that, for the moment, the approximation parameter is not taken into account. We consider two generalizations of the Szász operator, defined as follows. Let $X_1, X_2, \ldots$ be a sequence of independent and identically distributed non-negative random variables. We define the operators

$$Mf(t) = \sum_{k=0}^{\infty} Ef\left(\sum_{i=1}^{k+1} X_i\right)p_t(k), \qquad M^*f(t) = \sum_{k=0}^{\infty} Ef\left(\sum_{i=1}^{k} X_i\right)p_t(k), \tag{4}$$

where, in the previous formula, E denotes mathematical expectation. Observe that the Szász operator is a particular case of the operator M^*, by taking $X_i = 1$. On the other hand, if the X_i's are absolutely continuous random variables with probability density function d, the previous operators can be written as

$$Mf(t) = \int_0^{\infty} f(u)h_t(u)\,du; \qquad M^*f(t) = p_t(0)f(0) + \int_0^{\infty} f(u)h_t^*(u)\,du \tag{5}$$

where

$$h_t(u) = \sum_{k=0}^{\infty} p_t(k)d_{k+1}(u) \quad \text{and} \quad h_t^*(u) = \sum_{k=1}^{\infty} p_t(k)d_k(u).$$

Here, d_k is the convolution of d with itself k times. If the X_i's are exponential random variables with mean 1 (that is, having density g_1, as defined in (1)), these operators were introduced in [23] as a Durrmeyer-type modification of the Szász operator. Some of their preservation properties were analyzed in [2] using probabilistic methods.

Observe that the previous operators (and some other Bernstein-type operators) can be built using the following probabilistic representation. Let $(X(t),\ t \geqslant 0)$ be a stochastic process. Consider an operator of the form

$$Tf(t) = Ef\big(X(t)\big), \quad t \geqslant 0,\ f \in \mathcal{T}, \tag{6}$$

in which $\mathcal{T}$ is the set of measurable functions $f : [0,\infty) \to \mathbb{R}$ such that $E|f|(X(t)) < \infty$, $t \geqslant 0$.

In order to build a sequence of smooth functions $T_r f(t)$ approximating f as r tends to infinity, one considers

$$T_r f(t) = E\left[f\left(\frac{X(tr)}{r}\right)\right], \quad r > 0. \tag{7}$$

In particular, if the underlying process satisfies $E[X(t)] = t$ (in other words, the operator is centered) and belongs to the class of Lévy processes (independent and stationary increments, cf. [26], for instance), we can guarantee that

$$\lim_{r\to\infty} T_r f(t) = f(t), \quad t \geqslant 0, \tag{8}$$

for all bounded and continuous f (this is a consequence of the strong law of large numbers for Lévy processes [26, Theorem 36.5, p. 246]).

Thus, coming back to the Szász operator (2), we have the following. Let $(N(t),\ t \geqslant 0)$ be a standard Poisson process (in particular, it is a Lévy process with $N(0) = 0$ and for $t > 0$ each $N(t)$ has Poisson distribution with probability mass function given by $p_t(k)$, as defined in (1)). Then,

$$Lf(t) = Ef\big(N(t)\big), \qquad L_r f(t) = E\left[f\left(\frac{N(tr)}{r}\right)\right]. \tag{9}$$

With respect to the Gamma type operator (3), we consider a standard Gamma process $(S(t),\ t \geqslant 0)$ (that is, a Lévy process with $S(0) = 0$ and each $S(t)$ has Gamma density g_t, as defined in (1)). Then,