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Existence and global asymptotic behavior of positive solutions for nonlinear problems on the half-line

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ABSTRACT

In this paper, we aim at studying the existence, uniqueness and the exact asymptotic behavior of positive solutions to the following boundary value problem

$$\begin{cases} \dfrac{1}{A}(Au')' + a(t)u^{\sigma} = 0, \quad t \in (0,\infty), \\ \lim\limits_{t\to 0^+} u(t) = 0, \quad \lim\limits_{t\to\infty} \dfrac{u(t)}{\rho(t)} = 0, \end{cases}$$

where $\sigma < 1$, A is a continuous function on $[0,\infty)$, positive and differentiable on $(0,\infty)$ such that $\frac{1}{A}$ is integrable on $[0,1]$ and $\int_0^\infty \frac{1}{A(t)}\,dt = \infty$. Here $\rho(t) = \int_0^t \frac{1}{A(s)}\,ds$, for $t \geqslant 0$ and a is a nonnegative continuous function that is required to satisfy some assumptions related to the Karamata classes of regularly varying functions. Our arguments are based on monotonicity methods.

1. Introduction

In [15], Zhao considered the following problem

$$\begin{cases} u'' + \varphi(.,u) = 0, \quad \text{on } (0,\infty), \\ u > 0, \quad \text{on } (0,\infty), \\ \lim\limits_{t\to 0^+} u(t) = 0, \end{cases} \tag{1.1}$$

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where φ is a measurable function on $(0,\infty)\times(0,\infty)$, dominated by a convex positive function. Then he showed that there exists $b>0$ such that for each $\mu\in(0,b]$, there exists a positive continuous solution u of (1.1) satisfying $\lim_{t\to\infty}\frac{u(t)}{t}=\mu$.

On the other hand, in [10], Mâagli and Masmoudi generalized the result of Zhao to the more general boundary value problem

$$\begin{cases} \frac{1}{A}\big(Au'\big)'+f\big(.,u,Au'\big)=0, & \text{on } (0,\infty),\\ u>0, & \text{on } (0,\infty),\\ \lim\limits_{t\to 0^+}u(t)=0, \end{cases} \tag{1.2}$$

where A is a positive and differentiable function on $(0,\infty)$ and f is a measurable function on $(0,\infty)\times(0,\infty)\times(0,\infty)$, which may change sign and is dominated by a regular function. Then they proved the existence of a constant $b>0$ such that for each $\mu\in(0,b]$, problem (1.2) has a continuous solution u satisfying $\lim_{t\to\infty}\frac{u(t)}{\rho(t)}=\mu$, where $\rho(t)=\int_0^t\frac{1}{A(s)}\,ds$, for $t\geqslant 0$.

Note also that various existence results for this type of equations have appeared in the literature (see [1–15] and the references therein).

In this paper, we aim at studying the existence, uniqueness and the exact asymptotic behavior of positive solution to the following boundary value problem

$$\begin{cases} \frac{1}{A}\big(Au'\big)'+a(t)u^{\sigma}=0, & t\in(0,\infty),\\ u>0, & \text{on } (0,\infty),\\ \lim\limits_{t\to 0^+}u(t)=0, & \lim\limits_{t\to\infty}\frac{u(t)}{\rho(t)}=0, \end{cases} \tag{1.3}$$

where $\sigma<1$, A is a continuous function on $[0,\infty)$, positive and differentiable on $(0,\infty)$. We also assume that $\frac{1}{A}$ is integrable on $[0,1]$ and $\int_0^\infty\frac{1}{A(t)}\,dt=\infty$. The function ρ is defined by $\rho(t)=\int_0^t\frac{1}{A(s)}\,ds$, for $t\geqslant 0$.

The nonnegative potential function a is required to be continuous on $(0,\infty)$ that may be singular at 0 or unbounded near ∞ and satisfying some conditions related to the Karamata classes $\mathcal{K}$ and $\mathcal{K}^\infty$ (see Definitions 1.1 and 1.2 below).

For the case $\sigma<0$, the existence and the uniqueness of a positive continuous bounded solution to problem (1.3) is proved in [2, Theorem 2], under the condition that a is a positive continuous function on $(0,\infty)$ satisfying

$$\int_0^\infty A(s)\min\big(1,\rho(s)\big)a(s)\,ds<\infty. \tag{1.4}$$

Also some estimates for such solution are given. Thus, it is interesting to know the exact asymptotic behavior and to extend the study of (1.3) to $0\leqslant\sigma<1$.

Throughout this paper and without loss of generality, we assume that $\int_0^1\frac{1}{A(t)}\,dt=1$.

To state our result, we need some notations. We first introduce the following Karamata classes of regularly varying functions.

Definition 1.1. The class $\mathcal{K}$ is the set of all Karamata functions L defined on $(0,\eta]$ by

$$L(t):=c\exp\left(\int_t^\eta\frac{z(s)}{s}\,ds\right),$$

for some $\eta>1$ and where $c>0$ and $z\in C([0,\eta])$ such that $z(0)=0$.