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Existence of a lower bound for the distance between point masses of relative equilibria in spaces of constant curvature

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ABSTRACT

We prove that if for the curved n-body problem the masses are given, the minimum distance between the point masses of a specific type of relative equilibrium solution to that problem has a universal lower bound that is not equal to zero. We furthermore prove that the set of all such relative equilibria is compact. This class of relative equilibria includes all relative equilibria of the curved n-body problem in $\mathbb{H}^2$ and a significant subset of the relative equilibria for $\mathbb{S}^2$, $\mathbb{S}^3$ and $\mathbb{H}^3$.

1. Introduction

By n-body problems, we mean problems where we want to find the dynamics of n point particles. By relative equilibria, we mean solutions to such problems where the point particles represent rotating configurations of fixed size and shape. The *n-body problem in spaces of constant curvature*, or *curved n-body problem* is an extension of the Newtonian n-body problem (in Euclidean space) into spaces of nonzero, constant Gaussian curvature, which means that the space is either spherical (if the curvature is positive), or hyperbolical (if the curvature is negative) (see [11,9,10]). It was noted in [5] and [7] that it suffices to consider the case that the curvature is equal to either +1, or −1. More precisely, following [11,9,10,6,7], if we define the space

$$\mathbb{M}^k_\sigma = \left\{(x_1,\ldots,x_{k+1}) \in \mathbb{R}^{k+1} \;\middle|\; x_1^2+\cdots+x_k^2+\sigma x_{k+1}^2=\sigma\right\},$$

where σ equals either +1, or −1 and for x, $y \in \mathbb{M}^k_\sigma$ define the inner product

$$x \odot_k y = x_1y_1+\cdots+x_ky_k+\sigma x_{k+1}y_{k+1},$$

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we mean the problem of finding the dynamics of n point particles with respective masses $m_1, \ldots, m_n$ and coordinates $q_1, \ldots, q_n \in \mathbb{M}_\sigma^k$, $k \geqslant 2$, as described by the system of differential equations

$$\ddot{q}_i = \sum_{j=1, j \neq i}^{n} \frac{m_j(q_j - \sigma(q_i \odot_k q_j)q_i)}{(\sigma - \sigma(q_i \odot_k q_j)^2)^{\frac{3}{2}}} - \sigma(\dot{q}_i \odot_k \dot{q}_i)q_i, \quad i \in \{1, \ldots, n\}. \tag{1.1}$$

The first to investigate n-body problems for spaces of constant curvature were Bolyai [1] and Lobachevsky [19], who independently proposed a curved 2-body problem in hyperbolic space $\mathbb{H}^3$ in the 1830s. Since then, n-body problems for spaces of constant curvature have been studied by mathematicians such as Dirichlet, Schering [21,20], Killing [12–14], Liebmann [16–18] and more recently Kozlov and Harin [15] and Cariñena, Rañada and Santander [2]. However, the study of n-body problems in spaces of constant curvature for the case that $n \geqslant 2$ started with [11,9,10] by Diacu, Pérez-Chavela and Santoprete. After this breakthrough, additional results for the $n \geqslant 2$ case were then obtained by Diacu [3,4,6], Diacu, Kordlou [7], Diacu, Pérez-Chavela [8]. For a more detailed historical overview, please see [4,6,5,7], or [11].

M. Shub proved for the Newtonian n-body problem that if we fix the masses and the angular velocity (i.e. the speed with which the angle of the rotation changes), the set of possible relative equilibria is compact and as a direct consequence that there exists a universal nonzero lower bound for the distance between the point particles of the relative equilibria in such a set (see [22]). Shub's results were a potential first step in what may lead to a proof of the famous sixth Smale problem (see [23]) which states that such sets are not only compact, but, in fact, finite.

In this paper, following Shub's line of thought, we will make a first attempt at investigating to which extent we can extend his results to the constant curvature case. More specifically, let

$$T(t) = \begin{pmatrix} \cos(t) & -\sin(t) \\ \sin(t) & \cos(t) \end{pmatrix}$$

be a 2×2 rotation matrix, $A > 0$, $Q_1, \ldots, Q_n \in \mathbb{R}^2$ and $Z \in \mathbb{R}^{k-1}$ constant. Then we will call any solution $q_1, \ldots, q_n$ of (1.1) of the form

$$q_i(t) = \begin{pmatrix} T(At)Q_i \\ Z \end{pmatrix}, \quad i \in \{1, \ldots, n\}, \tag{1.2}$$

a *relative equilibrium* and A its *angular velocity*. Let $\|\cdot\|_p$ be the Euclidean norm on $\mathbb{R}^p$. Let $\epsilon > 0$ and let R_ϵ be the set of all relative equilibria in $\mathbb{S}^k$ for which $\|Z\|_{k-1} > \epsilon$ together with all relative equilibria in $\mathbb{H}^k$. We will prove that

Theorem 1.1. *There exists a universal constant $C > 0$ such that for any relative equilibrium solution in R_ϵ of (1.1) $\|q_i - q_j\|_k > C$ for all $i, j \in \{1, \ldots, n\}$, $i \neq j$ if the masses $m_1, \ldots, m_n$ are given.*

and

Theorem 1.2. *If we write any set of vectors $Q_1, \ldots, Q_n \in \mathbb{R}^2$ of a relative equilibrium solution $q_1, \ldots, q_n$ in R_ϵ as a $2n$-dimensional vector*

$$\begin{pmatrix} Q_1 \\ \vdots \\ Q_n \end{pmatrix},$$

then the set of all such $2n$-dimensional vectors, for fixed masses $m_1, \ldots, m_n$ and angular velocity A, is compact in $\mathbb{R}^{2n}$.