J. Math. Anal. Appl. 416 (2014) 229–257

Contents lists available at ScienceDirect

Journal of Mathematical Analysis and Applications

www.elsevier.com/locate/jmaa

Rotationally symmetric p-harmonic flows from D^2 to S^2: Local well-posedness and finite time blow-up

Razvan Gabriel Iagar [a,b,*], Salvador Moll [a]

[a] Departamento de Análisis Matemático, Univ. de Valencia, Dr. Moliner 50, 46100, Burjassot (Valencia), Spain
[b] Institute of Mathematics of the Romanian Academy, P.O. Box 1-764, RO-014700, Bucharest, Romania

ARTICLE INFO

Article history:
Received 18 June 2013
Available online 25 February 2014
Submitted by V. Radulescu

Keywords:
p-harmonic flow
Rotational symmetry
p-Laplacian
Local well-posedness
Finite time blow-up
Image processing
Liquid crystals
Ferromagnetism

ABSTRACT

We study the p-harmonic flow from the unit disk D^2 to the unit sphere S^2 under rotational symmetry. We show that the Dirichlet problem with constant boundary condition is locally well-posed in the class of classical solutions and we also give a sufficient criterion, in terms of the boundary condition, for the derivative of the solutions to blow-up in finite time.

1. Introduction and main results

In this paper we will focus on the following system of PDEs

$$\mathbf{u}_t = \operatorname{div}\big(|\nabla \mathbf{u}|^{p-2}\nabla \mathbf{u}\big) + \mathbf{u}|\nabla \mathbf{u}|^p, \tag{1.1}$$

where $p > 1$ and $\mathbf{u} : \Omega \mapsto S^N$, with Ω being an open domain in $\mathbb{R}^N$ and S^N the unit sphere of $\mathbb{R}^{N+1}$. This system of PDEs is called the *p-harmonic flow* and it corresponds to the gradient flow in $L^2(\Omega)$ of the following energy functional:

$$E_p(\mathbf{u}) := \frac{1}{p}\int_{\Omega} |\nabla \mathbf{u}|^p \, dx. \tag{1.2}$$

* Corresponding author.
E-mail addresses: razvan.iagar@uv.es (R.G. Iagar), j.salvador.moll@uv.es (S. Moll).

http://dx.doi.org/10.1016/j.jmaa.2014.02.045

Apart from the interest on the results of the competition between the two effects appearing in the system (diffusion and reaction), (1.1) arises from a number of applications: ferromagnetism [6], theory of liquid crystals [12], multigrain problems [15] and image processing [18].

We study the Dirichlet problem for (1.1) with boundary condition $\mathbf{u}(t,x) = \mathbf{u}_0(x)$ for $(t,x) \in \partial Q$, where $Q := (0,\infty) \times \Omega$. Related to this problem, there are several previous results for the semilinear diffusion case $p = 2$, as in [2,3], and also for $p > 1$ [4,7,13,17]. A special case in the analysis, with some particular mathematical features due to lack of sufficient regularity, is the limit case $p = 1$, intensively studied recently; indeed, local well-posedness, steady states and either the appearance of finite time blow-up or the existence of global solutions were studied under the hypothesis of rotational symmetry in [5,10,11]; in particular, the most interesting feature that has been found is the occurrence of blow-up in finite time of the derivative. The Neumann problem for $p = 1$ has been investigated recently in [8,9].

In a previous work [14], the authors studied the rotationally symmetric steady states corresponding to the system for $1 < p < \infty$, $M = S^2$, the unit sphere in $\mathbb{R}^3$, $\Omega = D^2$, the unit disk in $\mathbb{R}^2$. A rotationally symmetric solution to (1.1) has the general form

$$\mathbf{u}(t,x) = \left(\frac{x_1}{r}\sin h(t,r), \frac{x_2}{r}\sin h(t,r), \cos h(t,r)\right), \quad r = |x|,\ x = (x_1,x_2) \in D^2, \tag{1.3}$$

and the Dirichlet boundary condition $h(t,1) = h(0,1) = l$. The energy functional E_p in (1.2) becomes

$$E_p(\mathbf{u}) = \int_0^1 L_p(r,h,h_r)\,dr, \tag{1.4}$$

with a non-coercive Lagrangian given by

$$L_p(r,s,\xi) = \left(r^{2/p}h_r^2 + r^{(2-2p)/p}\sin^2 h\right)^{p/2}.$$

By replacing $\mathbf{u}$ in (1.1) by its special form (1.3), the system (1.1) becomes

$$\begin{aligned} h_t = \left(h_r^2 + \frac{sin^2 h}{r^2}\right)^{(p-4)/2} \bigg[&(p-1)h_r^2 h_{rr} + (p-3)\left(\frac{h_r^2 \sin h\cos h}{r^2} - \frac{h_r \sin^2 h}{r^3}\right) \\ &+ \frac{h_r^3}{r} + \frac{h_{rr}\sin^2 h}{r^2} - \frac{\sin^3 h\cos h}{r^4}\bigg], \end{aligned} \tag{1.5}$$

see [14, p. 3930] for details. In our previous work, we studied and classified the steady states for (1.5), for all $1 < p < \infty$.

In the present paper we continue the study of the rotationally symmetric flows begun in [14], and we devote it to some qualitative properties of the solution $\mathbf{u}$ to (1.1), or equivalently, h solution to (1.5), such as existence and well-posedness for short times, regularity of solutions and conditions for the phenomenon of blow-up in finite time to occur. We restrict ourselves to the fast diffusion range $1 < p < 2$, since, for the range $p \geqslant 2$, it has been shown in [7] that strong solutions exist globally for any initial datum, they are Hölder continuous and they converge to a steady-state as $t \to +\infty$. A similar result for the range $1 < p < 2$ was also obtained in [7] but with a condition of small energy on the initial datum. We point out that our approach and results are totally different to those in [7]. First of all, we obtain local existence of classical solutions for any initial datum and secondly, we give a sufficient condition on the boundary constraint for classical solutions ceasing to exist in finite time.