J. Math. Anal. Appl. 416 (2014) 374–389

Contents lists available at ScienceDirect

Journal of Mathematical Analysis and Applications

www.elsevier.com/locate/jmaa

Nonuniform continuity of the solution map to the two component Camassa–Holm system

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ARTICLE INFO

Article history:
Received 28 September 2013
Available online 28 February 2014
Submitted by J. Shi

Keywords:
Two-component Camassa–Holm system
Cauchy problem
Nonuniform continuous

ABSTRACT

We prove that the solution map of the two-component Camassa–Holm system is not uniformly continuous as a map from a bounded subset of the Sobolev space $H^s(\mathbb{T}) \times H^r(\mathbb{T})$ to $C([0,1], H^s(\mathbb{T}) \times H^r(\mathbb{T}))$ when $s \geqslant 1$ and $r \geqslant 0$. We also demonstrate the nonuniform continuous property in the continuous function space $C^1(\mathbb{T}) \times C^1(\mathbb{T})$.

1. Introduction

The Camassa–Holm (CH) equation

$$u_t + uu_x + \left(1 - \partial_x^2\right)^{-1}\left(u^2 + \frac{1}{2}(u_x)^2\right)_x = 0$$

was first obtained by Fokas and Fuchssteiner [10] as a bi-Hamiltonian system with infinitely many conservation laws then by Camassa and Holm [1] as a completely integrable model for shallow-water waves. Recently a rigorous justification of the derivation of the CH equation as an approach to the governing equations for water waves was provided by Constantin and Lannes [6]. The CH equation admits peaked solitary waves or "peakons" [1,21]: $u(t,x) = ce^{-|x-ct|}$, $c \neq 0$, which are smooth except at the crests, where they are continuous, but have a jump discontinuity in the first derivative. The peakons capture a feature that is characteristic for the waves of great height – waves of the largest amplitude that are exact solutions of the governing equations for water waves [1,4,6,7]. The CH equation also models wave breaking (i.e. the solution remains bounded while its slope becomes unbounded in finite time) [4,6,7]. See [21,22,25] and the references therein for more properties of the CH equation.

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http://dx.doi.org/10.1016/j.jmaa.2014.02.047
0022-247X/

In this paper we consider the following generalized two-component CH system

$$\begin{cases} u_t + uu_x + (1-\partial_x^2)^{-1}\left(u^2 + \frac{1}{2}(u_x)^2 + \frac{1}{2}\rho^2\right)_x = 0, \\ \rho_t + (u\rho)_x = 0 \end{cases} \tag{1.1}$$

established by Chen, Liu and Zhang [2] that there is a reciprocal transformation between the system and the first negative flow of the AKNS hierarchy. System (1.1) was also rigorously justified by Constantin and Ivanov [5] to approximate the governing equations for shallow water waves, where the variable $u(t,x)$ represents the horizontal velocity of the fluid, and $\rho(t,x)$ is related to the free surface elevation from equilibrium (or scalar density). More recently, Holm, Nraigh and Tronci [15] proposed a modified two-component Camassa–Holm system which possesses singular solutions in component ρ. Clearly if $\rho(t,x) \equiv 0$, system (1.1) returns to the well-known CH equation. The two-component CH system is completely integrable [9,16] as it can be written as a compatibility condition of two linear systems (Lax pair) with a spectral parameter ζ, that is,

$$\begin{aligned} \Psi_{xx} &= \left(-\zeta^2\rho^2 + \zeta m + \frac{1}{4}\right)\Psi, \\ \Psi_t &= \left(\frac{1}{2\zeta} - u\right)\Psi_x + \frac{1}{2}u_x\Psi, \end{aligned}$$

and has a bi-Hamiltonian structure corresponding to the Hamiltonian

$$H_1 = \frac{1}{2}\int \left(mu + (\rho-1)^2\right) dx$$

with $m = u - u_{xx}$ and the Hamiltonian

$$H_2 = \frac{1}{2}\int \left(u(\rho-1)^2 + 2u(\rho-1) + u^3 + uu_x^2\right) dx.$$

There are two Casimirs, i.e. $\int(\rho-1)$ and $\int m$ with periodic boundary conditions, or $u \to 0$ and $\rho \to 1$ as $|x| \to \infty$.

Mathematical properties of (1.1) have been studied in many works. For example, Escher, Lechtenfeld and Yin [8] investigated local well-posedness for the two-component CH system with initial data $(u_0, \rho_0 - 1) \in H^s(\mathbb{R}) \times H^{s-1}(\mathbb{R})$ with $s \geqslant 2$ and derived some precise blow-up scenarios for strong solutions to the system. Constantin and Ivanov [5] provided some conditions of wave breaking and small global solutions. Gui and Liu [11] obtained results of local well-posedness in the Besov spaces and wave breaking for certain initial profiles. Zhang and Liu [26] studied the stability of solitary waves for the two-component CH system.

What we will investigate here is the issue related to the well-posedness of (1.1), namely the continuous dependence on the initial data. It is known that if the local well-posedness of solutions to a certain evolutionary PDE can be established by a solely fixed point theorem for contraction mappings, then the data-to-solution map will be Lipschitz on the space where solutions live. For instance the KdV equation is shown to be well-posed in $H^s(\mathbb{R})$ with $s > -3/4$ and the solution map is Lipschitz on the same $H^s(\mathbb{R})$ [18]. One of the key ingredients is the Strichartz estimates achieved from the strong dispersive effect.

When there is no dispersion, cf. the Burgers equation, however, Kato [17] proved the local well-posedness in $H^k(\mathbb{R})$ for any integer $k \geqslant 2$ and showed that the dependence on initial data in $H^k(\mathbb{R})$ is continuous but not Hölder continuous with any prescribed exponent. The phenomenon of not uniform continuity for some dispersive equations can be found, for example in Kenig, Ponce and Vega [19].

The motivation for this paper comes from the work of Himonas and Misiołek [13] where non-uniform dependence of the solution map for the periodic CH equation on Sobolev spaces $H^s(\mathbb{T}), s \geqslant 2$, is proved