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Unitary operators in the orthogonal complement of a type I von Neumann subalgebra in a type II_1 factor ☆

Rui Shi *, Xiaoyan Zhou

School of Mathematical Sciences, Dalian University of Technology, Dalian 116024, China

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ABSTRACT

It is well-known that the equality

$$L_G \ominus L_H = \overline{\operatorname{span}\{L_g\colon\ g \in G - H\}}^{\mathrm{SOT}}$$

holds for G an i.c.c. group and H a subgroup in G, where L_G and L_H are the corresponding group von Neumann algebras and $L_G \ominus L_H$ is the set $\{x \in L_G\colon E_{L_H}(x) = 0\}$ with E_{L_H} the conditional expectation defined from L_G onto L_H. Inspired by this, it is natural to ask whether the equality

$$N \ominus A = \overline{\operatorname{span}\{u\colon\ u \text{ is unitary in } N \ominus A\}}^{\mathrm{SOT}}$$

holds for N a type II_1 factor and A a von Neumann subalgebra of N. In this paper, we give an affirmative answer to this question for the case A a type I von Neumann algebra.

1. Introduction

Throughout this paper, all Hilbert spaces discussed are *complex and separable*. Let (N, τ) be a finite von Neumann algebra with a faithful normal normalized trace τ and A be a von Neumann subalgebra of N. Then the trace τ induces an inner product $\langle\cdot,\cdot\rangle$ on N which is defined by $\langle x, y\rangle = \tau(y^*x)$, $\forall x, y \in N$. Denote by $L^2(N)$ (resp. $L^2(A)$) the completion of N (resp. A) with respect to the inner product, then $L^2(A)$ is a subspace of $L^2(N)$. Let e_A denote the projection from $L^2(N)$ onto $L^2(A)$. The trace-preserving conditional expectation E_A of N onto A is defined to be the restriction $e_A|_N$. By [2], E_A has the following properties:

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* Corresponding author.
E-mail addresses: ruishi.math@gmail.com (R. Shi), xiaoyanzhou.oa@gmail.com (X. Zhou).

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(1) $e_A|_N = E_A$ is a norm reducing map from N onto A with $E_A(1) = 1$;
(2) the equality $E_A(axb) = aE_A(x)b$ holds for all $x \in N$ and $a, b \in A$;
(3) $\tau(xE_A(y)) = \tau(E_A(x)E_A(y)) = \tau(E_A(x)y)$ for all $x, y \in N$;
(4) $e_A x e_A = E_A(x)e_A = e_A E_A(x)$ for all $x \in N$.

Let G be a (countable) discrete i.c.c. group and denote by $l^2(G)$ the Hilbert space of square-summable sequences. Given every g in G, the operator L_g is defined by $(L_g x)(h) = x(g^{-1}h)$, for every x in $l^2(G)$ and h in G. This is a unitary operator. Let L_G be the von Neumann algebra generated by $\{L_g\colon\ g \in G\}$. It is well-known that L_G is a type II_1 factor. For a subgroup H in G, define

$$L_G \ominus L_H \triangleq \bigl\{x \in L_G\colon\ E_{L_H}(x) = 0\bigr\}.$$

Thus we obtain that

$$L_G \ominus L_H = \overline{\operatorname{span}\{L_g\colon\ g \in G - H\}}^{\mathrm{SOT}}.$$

Inspired by this, it is natural to ask whether the equality

$$N \ominus A = \overline{\operatorname{span}\{u\colon\ u \text{ is unitary in } N \ominus A\}}^{\mathrm{SOT}}$$

holds for N a type II_1 factor and A a von Neumann subalgebra of N. In this paper, we give an affirmative answer to this question for the case A a type I von Neumann algebra in Theorem 2.6.

In [1], A. Ioana, J. Peterson and S. Popa proved a series of rigidity results for amalgamated free product II_1 factors, which can be viewed as von Neumann algebra versions of the "subgroup theorems" and "isomorphism theorems" for amalgamated free product groups in Bass–Serre theory. They introduced the concept "bounded homogeneous orthonormal basis" of M over B, where (M, τ) is a separable finite von Neumann algebra and $B \subset M$ is a von Neumann subalgebra. In the current paper, our result indicates that it is possible to choose unitary elements to form a bounded homogeneous orthonormal basis with respect to a type I von Neumann subalgebra of M.

2. Proofs

In this paper, the matrix representations of operators will be used frequently. We briefly recall the relationship between conditional expectations with respect to matrix representations of operators. Let $e_1, \ldots, e_n \in N$ be mutually equivalent orthogonal projections such that $\sum_{i=1}^n e_i = 1$, where 1 is the identity of N. Then for every $x \in N$, we can express x in the form

$$x = \begin{pmatrix} x_{11} & \cdots & x_{1n} \\ \vdots & \ddots & \vdots \\ x_{n1} & \cdots & x_{nn} \end{pmatrix} \begin{matrix} \operatorname{ran} e_1 \\ \vdots \\ \operatorname{ran} e_n \end{matrix}$$

and there exists a $*$-isomorphism φ from N onto $\mathbb{M}_n(N_{e_1})$, where N_{e_1} is the restriction of $e_1 N e_1$ on $\operatorname{ran} e_1$ and denote by $\mathbb{M}_n(N_{e_1})$ the set n-by-n matrices with entries in N_{e_1}. On the other hand, let τ be a faithful normal normalized trace on N, and the trace τ_n on $\mathbb{M}_n(N)$ is defined by $\tau_n(x) = \frac{1}{n}(\sum_{i=1}^n \tau(x_{ii}))$, where x in $\mathbb{M}_n(N)$ is of the form

$$x = \begin{pmatrix} x_{11} & \cdots & x_{1n} \\ \vdots & \ddots & \vdots \\ x_{n1} & \cdots & x_{nn} \end{pmatrix}$$