J. Math. Anal. Appl. 414 (2014) 99–109

Contents lists available at ScienceDirect

Journal of Mathematical Analysis and Applications

www.elsevier.com/locate/jmaa

On the Cauchy problem for a class of iterated Fuchsian partial differential equations

Mohamed Amine Kerker

Laboratoire de Mathématiques, EA 4535, Université de Reims Champagne-Ardenne, Moulin de la Housse, B.P. 1039, F-51687, Reims Cedex 2, France

ARTICLE INFO

Article history:
Received 17 December 2012
Available online 30 December 2013
Submitted by T. Witelski

Keywords:
Fuchsian partial differential equations
Gauss hypergeometric function
Iterated differential equations
Singularities

ABSTRACT

In this paper, we consider the Cauchy problem with ramified data for a class of iterated Fuchsian partial differential equations. We give an explicit representation of the solution in terms of Gauss hypergeometric functions. Our results are illustrated through some examples.

1. Introduction

In this paper, we consider, in a neighborhood of the origin of $\mathbb{C}^2$, a class of Fuchsian partial differential equations

$$L_{\alpha_1} \dots L_{\alpha_n} u = 0, \tag{1}$$

subject to the singular initial conditions

$$\partial^i u(0,x) = u_i(x), \quad i = \overline{1,n},$$

where α_i, $i = \overline{1,n}$, are complex constants, and where L_{α_i} is the second order Fuchsian partial differential operator:

$$L_{\alpha_i} := t\partial_t^2 - \partial_x^2 + \alpha_i \partial_t.$$

The iterated operators $L_{\alpha_1} \dots L_{\alpha_n}$ are defined by the relations

$$L_{\alpha_1} \dots L_{\alpha_n} u = L_{\alpha_1} \dots L_{\alpha_{n-1}}(L_{\alpha_n} u).$$

0022-247X/$ – see front matter
http://dx.doi.org/10.1016/j.jmaa.2013.12.051

In [15] Weinstein studied the general solution of a class of iterated PDE of even order which includes iterated wave equations and iterated Euler–Poisson–Darboux equations. Using two recursion formulae, he decomposed the general solution into a sum of two functions each satisfying an equation of the second order. Since then, many iterated equations were studied [2,7,9,11].

Notice that Eq. (1) corresponds to some of the well-known classical equations and their iterated forms. Indeed, the equation $L_\alpha u = 0$ is equivalent to the Euler–Poisson–Darboux equation,

$$\partial_y^2 u - \partial_x^2 u + \frac{2\alpha - 1}{y}\partial_y u = 0,$$

by performing the change of variables $y = 2\sqrt{t}$; the wave equation is then obtained by taking $\alpha = \frac{1}{2}$. Also, the change of variables $y = t^{1-\alpha}$, $\alpha \neq 1$, in $L_\alpha u = 0$, leads to the Tricomi type equation

$$\partial_y^2 u - \frac{y^{2\alpha-1}}{(\alpha-1)^2}\partial_x^2 u = 0.$$

Eq. (1) is a Fuchsian partial differential equation of weight n in the sense of Baouendi and Goulaouic [3]. Hence, only n initial conditions (rather than $2n$) suffice to determine the unique solution, regular on the initial surface, of our Cauchy problem.

The Fuchsian PDE is a natural extension of ordinary differential equations with regular singularity at a point. For such equations, M.S. Baouendi and C. Goulaouic [3] established fundamental results that are generalizations of the Cauchy–Kowalevsky theorem and the Holmgren theorem. Over the last four decades, Fuchsian PDEs have received intense attention in various settings, see for instance [5,10,12,14].

In this work, we aim to study the singularities of the solutions of our Cauchy problem. Generally it is difficult to investigate the properties of the singularities of solutions for PDEs. A natural approach is to seek the explicit representation of the solutions which makes the study of their singularities easier. Our method is to construct solutions in terms of Gauss hypergeometric functions (GHFs for short). Since it has intrinsic singularities, the GHF was used successfully, in many works, to construct the explicit solutions and then study their singularities, see e.g., [4–6,8,14] and references therein.

In our work, using an appropriate recursion formula, we give first the general solution of the iterated equations (1), and then choose those satisfying ramified initial conditions. Our purpose is to represent solutions by means of GHFs, which permit to us to study the singularities and the analytic continuation. We focus on the study of the Cauchy problem for Eq. (1) when $n = 2$, the general problem is treated similarly.

2. Preliminary results

We note by $u^{(\alpha)}$ a solution of the equation $L_\alpha u = 0$. Then, the following fundamental recursion formulas hold:

Lemma 2.1.

1. $$u_t^{(\alpha)} = u^{(\alpha+1)}, \tag{2}$$

2. $$u^{(\alpha)} = t^{1-\alpha} u^{(2-\alpha)},$$