J. Math. Anal. Appl. 414 (2014) 243–249

Contents lists available at ScienceDirect

Journal of Mathematical Analysis and Applications

www.elsevier.com/locate/jmaa

On Young and Heinz inequalities for τ-measurable operators ☆

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A R T I C L E I N F O

Article history:
Received 2 July 2013
Available online 8 January 2014
Submitted by J.A. Ball

Keywords:
Von Neumann algebras
Positive operators
Young inequality
Heinz inequality

A B S T R A C T

The purpose of this article is to prove Young and Heinz inequalities for τ-measurable operators.

1. Introduction

Let $M_n(\mathbb{C})$ be the space of $n\times n$ complex matrices. Let $|||\cdot|||$ denote any unitarily invariant (or symmetric) norm on $M_n(\mathbb{C})$, that is to say, $|||UAV||| = |||A|||$ for all $A \in M_n(\mathbb{C})$ and for all unitary matrices U, $V \in M_n(\mathbb{C})$. In 1979, McIntosh [3] proved that for any unitary invariant norm Heinz inequality for matrices holds. Using the refinements of the classical Young inequality for positive real numbers, Kittaneh and Manasrah [2] established improved Young and Heinz inequalities for matrices.

In this paper we consider the noncommutative L^p-spaces of τ-measurable operators affiliated with a semi-finite von Neumann algebra equipped with a normal faithful semi-finite trace τ. We use the method of Kittaneh and Manasrah, via the notion of generalized singular value studied by Fack and Kosaki [1], to obtain generalizations of results in [2] for τ-measurable operators case.

2. Preliminaries

Throughout the paper we denote by $\mathcal{M}$ a semi-finite von Neumann algebra acting on the Hilbert space $\mathcal{H}$, with a normal faithful semi-finite trace τ. We denote the identity in $\mathcal{M}$ by 1 and let $\mathcal{P}$ denote the projection lattice of $\mathcal{M}$. A closed densely defined linear operator x in $\mathcal{H}$ with domain $D(x) \subseteq \mathcal{H}$ is said to be affiliated with $\mathcal{M}$ if $u^*xu = x$ for all unitary u which belong to the commutant $\mathcal{M}'$ of $\mathcal{M}$. If x is affiliated with $\mathcal{M}$,

☆ Partially supported by NSFC grant No. 11071204 and XJUBSCX-2012002.
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0022-247X/$ – see front matter
http://dx.doi.org/10.1016/j.jmaa.2014.01.017

then x is said to be τ-measurable if for every $\epsilon > 0$ there exists a projection $e \in \mathcal{M}$ such that $e(\mathcal{H}) \subseteq D(x)$ and $\tau(1-e) < \epsilon$. The set of all τ-measurable operators will be denoted by $L_0(\mathcal{M}, \tau)$, or simply $L_0(\mathcal{M})$. The set $L_0(\mathcal{M})$ is a $*$-algebra with sum and product being the respective closures of the algebraic sum and product. A closed densely defined linear operator x admits a unique polar decomposition $x = u|x|$, where u is a partial isometry such that $u^*u = (ker\, x)^{\perp}$ and $uu^* = \overline{im\, x}$ (with $im\, x = x(D(x))$). We call $r(x) = (ker\, x)^{\perp}$ and $l(x) = \overline{im\, x}$ the left and right supports of x, respectively. Thus $l(x) \sim r(x)$. Moreover, if x is self-adjoint, we let $s(x) = r(x)$, the support of x.

Let $\mathcal{M}_+$ be the positive part of $\mathcal{M}$. Set $\mathcal{S}_+(\mathcal{M}) = \{x \in \mathcal{M}_+ \colon \tau(s(x)) < \infty\}$ and let $\mathcal{S}(\mathcal{M})$ be the linear span of $\mathcal{S}_+(\mathcal{M})$, we will often abbreviate $\mathcal{S}_+(\mathcal{M})$ and $\mathcal{S}(\mathcal{M})$ respectively as $\mathcal{S}_+$ and $\mathcal{S}$. Let $0 < p < \infty$, the noncommutative L_p-space $L_p(\mathcal{M}, \tau)$ is the completion of $(\mathcal{S}, \|\cdot\|_p)$, where $\|x\|_p = \tau(|x|^p)^{\frac{1}{p}} < \infty$, $\forall x \in L_p(\mathcal{M}, \tau)$. In addition, we put $L^{\infty}(\mathcal{M}, \tau) = \mathcal{M}$ and denote by $\|\cdot\|_{\infty}(= \|\cdot\|)$ the usual operator norm. It is well known that $L_p(\mathcal{M}, \tau)$ are Banach spaces under $\|\cdot\|_p$ for $1 \leqslant p < \infty$ and they have a lot of expected properties of classical L_p-spaces (see [4] or [5]).

Let x be a τ-measurable operator and $t > 0$. The "t-th singular number (or generalized s-number) of x" is defined by

$$\mu_t(x) = \inf\bigl\{\|xe\| \colon\ e \in \mathcal{P},\ \tau(1-e) \leqslant t\bigr\}.$$

See [1] for basic properties and detailed information on the generalized s-numbers.

To achieve one of our main results, we state for easy reference the following fact obtaining from [6] that will be applied below.

Lemma 2.1. *Let $x, y \in L_p(\mathcal{M})$ be positive operators with $1 \leqslant p < \infty$ and let $z \in \mathcal{M}$, then*

$$\bigl\|x^v z y^{1-v}\bigr\|_p \leqslant \|xz\|_p^v \cdot \|zy\|_p^{1-v}, \quad 0 \leqslant v \leqslant 1.$$

3. Main results

First, we generalize the improved Young inequality in [2] for positive τ-measurable operators case.

Theorem 3.1. *Let $x, y \in L_1(\mathcal{M})$ be positive operators and let $0 \leqslant v \leqslant 1$, then*

$$\tau\bigl(x^v y^{1-v}\bigr) + r_0\bigl(\bigl(\tau(x)\bigr)^{\frac{1}{2}} - \bigl(\tau(y)\bigr)^{\frac{1}{2}}\bigr)^2 \leqslant \tau\bigl(vx + (1-v)y\bigr),$$

where $r_0 = \min\{v, 1-v\}$.

Proof. By Theorem 2.1 of [2] we have

$$v\mu_t(x) + (1-v)\mu_t(y) \geqslant \mu_t(x)^v \mu_t(y)^{1-v} + r_0\bigl(\mu_t(x)^{\frac{1}{2}} - \mu_t(y)^{\frac{1}{2}}\bigr)^2, \quad \forall\, t > 0.$$

Thus, together Lemma 4.2 of [1] with Hölder type inequality we get

$$\begin{aligned}
\tau\bigl(vx + (1-v)y\bigr) &= v\tau(x) + (1-v)\tau(y) \\
&= \int_0^{\infty} \bigl[v\mu_t(x) + (1-v)\mu_t(y)\bigr]\, dt \\
&\geqslant \int_0^{\infty} \mu_t(x)^v \mu_t(y)^{1-v}\, dt + r_0 \int_0^{\infty} \bigl[\mu_t(x) + \mu_t(y) - 2\mu_t\bigl(x^{\frac{1}{2}}\bigr)\mu_t\bigl(y^{\frac{1}{2}}\bigr)\bigr]\, dt
\end{aligned}$$