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Entropy of random chaotic interval map with noise which causes coarse-graining

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ABSTRACT

A random chaotic interval map with noise which causes coarse-graining induces a finite-state Markov chain. For a map topologically conjugate to a piecewise-linear map with the Lebesgue measure being ergodic, we prove that the Shannon entropy for the induced Markov chain possesses a finite limit as the noise level tends to zero. In most cases, the limit turns out to be strictly greater than the Lyapunov exponent of the original map without noise.

1. Introduction

For the study of random mapping dynamics, Lyapunov exponents play key roles. Furstenberg and Kesten [4] proved convergence of upper Lyapunov exponent for products of independent random matrices (see also Bougerol and Lacroix [2]). Diaconis and Freedman [3] proved almost sure convergence of the backward iteration if the random mapping is contracting on the average. Steinsaltz [12] proved almost sure convergence of the backward iteration for random logistic maps under the assumption that the averaged Lyapunov exponent is negative.

Matsumoto and Tsuda [8] observed that the numerical KS entropy for a modified BZ map with noise may fall below that for the original map without noise, and called this phenomenon the *noise-induced order*. For mathematical results, Sumi [13] proved that the chaos disappears for most of random complex dynamical systems for rational chaotic maps.

In order to study how a (non-random) mapping dynamics is affected by a noise, it may be useful to study how the Lyapunov exponent is related to some entropies for random chaotic maps. Araújo and Tahzibi [1] proved that the metric entropy of a random mapping dynamics, which was introduced by Kifer [6] via its skew product realization, falls below the KS entropy of the noise zero limit of the random mapping dynamics. Kozlov and Treshchev [7] and Piftankin and Treschev [11] proved that the coarse-graining Gibbs entropy converges to the KS entropy in the noise zero limit.

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In this paper, we study the noise zero limit of the entropy of random chaotic maps through an approach which is different from all the above results.

Let f be a chaotic map on the interval $[0,1]$ with invariant probability measure μ and consider a device which is designed to return $f(x)$ as output if input is x and if there is no noise. Suppose there is a noise which affects the device in such a way as coarse-graining the states; more precisely, the states are clustered into the set of subintervals $\Delta = \{A^{(1)}, \dots, A^{(N)}\}$ equivolume with respect to μ, and, if input is n taken from $\{1, \dots, N\}$, the device picks a point U from the subinterval $A^{(n)}$ at random with respect to μ conditional on $A^{(n)}$ and returns n' such that $f(U) \in A^{(n')}$ as output. To iterate this procedure independently induces a Markov chain taking values in $\{1, \dots, N\}$.

The purpose of this paper is to study the fine-graining limit as the noise level $1/N$ tends to zero of the Shannon entropy $H_\Delta(f)$ for the induced Markov chain. We shall prove that $\limsup H_\Delta(f)$ and $\liminf H_\Delta(f)$ are invariants with respect to topological conjugate. We shall also prove that, for piecewise-linear map with the Lebesgue measure being ergodic, the fine-graining limit does exist and is obtained explicitly. It is remarkable that the limit is always no less, and, in most cases strictly greater, than the Lyapunov exponent $\lambda(f)$ of the original (non-random) dynamical system (f, μ).

Let us give a small remark. Misiurewicz [9] and [10] studied continuity and discontinuity of topological entropies for piecewise monotone interval maps under perturbations preserving the number of pieces of monotonicity. He proved that the topological entropy for the skew tent maps is continuous. In a remarkable contrast, our fine-graining limit of the Shannon entropy for such a map is strictly greater than its Lyapunov exponent.

We give another small remark. The induced Markov chain can always be realized as a random mapping dynamics. So one may want to adopt the Shannon entropy of the random mapping dynamics rather than that of the Markov chain. However, the former is not less than the latter, and, in addition, the way of such realizations is not unique; see Yano and Yasutomi [14] and [15] for related results.

This paper is organized as follows. In Section 2, we prepare notations of the finite-state Markov chain induced by coarse-graining. In Section 3, we define $H_\Delta(f)$ and prove that its fine-graining limits are invariants with respect to topological conjugate. Section 4 is devoted to the computation of the fine-graining limit. In Section 5, we examine the results in the case of skew tent maps.

2. Random chaotic maps with noise which causes coarse-graining

Let $f : [0,1] \to [0,1]$ be a measurable map with a unique non-atomic invariant probability measure μ on $[0,1]$ which is ergodic. For a positive integer N, we call $\Delta = \{A^{(1)}, \dots, A^{(N)}\}$ an *equivolume partition* if Δ consists of disjoint subintervals of $[0,1]$ such that $\bigcup_{n=1}^{N} A^{(n)} = [0,1]$ and $\mu(A^{(n)}) = 1/N$ for $n = 1, \dots, N$. Since μ is non-atomic, the function $[0,1] \ni x \mapsto \mu([0,x]) \in [0,1]$ is continuous, so that there exists an equivolume partition. We write $\|\Delta\| = 1/N$, which will be called the *noise level*. Let $U = (U^{(1)}, \dots, U^{(N)})$ be a vector-valued random variable whose marginal $U^{(n)}$ is distributed as μ conditional on $A^{(n)}$, i.e.,

$$P\big(U^{(n)} \in B\big) = \frac{\mu(B \cap A^{(n)})}{\mu(A^{(n)})} = N\mu\big(B \cap A^{(n)}\big) \quad \text{for } B \in \mathcal{B}\big([0,1]\big). \tag{2.1}$$

We do not require any assumption for the joint distribution among $U^{(1)}, \dots, U^{(N)}$, because we only need the marginal distributions of U. Let $\pi^\Delta : [0,1] \to \{1, \dots, N\}$ be the projection map such that

$$\pi^\Delta[x] = n \quad \text{if and only if} \quad x \in A^{(n)}. \tag{2.2}$$

We define a random map f^Δ from $[0,1]$ to itself by

$$f^\Delta(x) = f\big(U^{(\pi^\Delta[x])}\big). \tag{2.3}$$