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Infinitely many solutions for a class of semilinear elliptic equations

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ABSTRACT

In this paper we find some new conditions to ensure the existence of infinitely many nontrivial solutions for the Dirichlet boundary value problems of the form $-\Delta u + a(x)u = g(x,u)$ in a bounded smooth domain. Conditions (S_1)–(S_3) in the present paper are somewhat weaker than the famous Ambrosetti–Rabinowitz-type superquadratic condition. Here, we assume that the primitive of the nonlinearity g is either asymptotically quadratic or superquadratic as $|u| \to \infty$.

1. Introduction and main results

In this work, we deal with the following elliptic boundary value problem:

$$\begin{cases} -\Delta u + a(x)u = g(x,u) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.1}$$

where $\Omega \subset \mathbb{R}^N$ $(N \geqslant 3)$ is bounded domain with smooth boundary $\partial\Omega$, $g \in \mathcal{C}(\bar{\Omega} \times \mathbb{R}, \mathbb{R})$ and $a \in \mathcal{L}^p(\Omega)$, $p > N/2$. Let $G(x,u) := \int_0^u g(x,s)\,\mathrm{d}s$, then G is of $\mathcal{C}^1$ class.

In the past decades, problem of the form (1.1) has been extensively studied, see for example, [1–12] and the references therein. For the case that G is of superquadratic growth, most of the results were obtained under the so-called Ambrosetti–Rabinowitz-type superquadratic condition (see, e.g., [1,3,7,8,10]). In recent papers the authors are interested in potentials satisfying conditions which are more general than (AR) condition (see, e.g., [5,6,11,12]). In this paper, we will study the existence of infinitely many nontrivial solutions of (1.1) via the new fountain theorems established in [12]. We divide the problem into the following two cases.

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1.1. The asymptotically quadratic case

Firstly we consider the asymptotically quadratic case. We make the following assumptions:

(A_1) $G(x,u) \geqslant 0$ for all $(x,u) \in \Omega \times \mathbb{R}$ and there exist constants $\mu \in (0,2)$ and $R_1 > 0$ such that $ug(x,u) \leqslant \mu G(x,u)$ for all $x \in \Omega$ and $|u| \geqslant R_1$;

(A_2) $\lim_{|u|\to 0} \frac{G(x,u)}{|u|^2} = \infty$ uniformly on Ω, and there exist constants $c_2, R_2 > 0$ such that $G(x,u) \leqslant c_2|u|$ for all $x \in \Omega$ and $|u| \leqslant R_2$;

(A_3) $\liminf_{|u|\to\infty} \frac{G(x,u)}{|u|} \geqslant d > 0$ uniformly in $x \in \Omega$.

Our first result reads as follows.

Theorem 1.1. *If (A_1)–(A_3) hold and $G(x,u)$ is even in u, then problem (1.1) possesses infinitely many solutions.*

1.2. The superquadratic case

Next, we consider the superquadratic situation.

(S_1) There exist constants $2 < p < 2^* = \frac{2N}{N-2}$ and $a_1 > 0$ such that

$$\left|g(x,u)\right| \leqslant a_1\left(1 + |u|^{p-1}\right), \quad \forall (x,u) \in \Omega \times \mathbb{R};$$

(S_2) $G(x,u) \geqslant 0$ for all $(x,u) \in \Omega \times \mathbb{R}$, and there exist constants $a_2, L_1 > 0$ and $q > 2$ such that $ug(x,u) \geqslant a_2|u|^q$ for all $x \in \Omega$ and $|u| \geqslant L_1$;

(S_3) there exist constants $\nu < \min\{q-1, N+q-\frac{N}{2}p\}$ and $a_3, L_2 > 0$ such that

$$ug(x,u) \geqslant \left(2 + \frac{a_3}{|u|^\nu}\right)G(x,u), \quad \forall x \in \Omega \text{ and } |u| \geqslant L_2.$$

We shall prove the following result.

Theorem 1.2. *If (S_1)–(S_3) hold and $G(x,u)$ is even in u, then problem (1.1) possesses infinitely many solutions.*

Remark 1.3. If we weaken the condition $q > 2$ in condition (S_2) as $q \geqslant 2$, and assume $G(x,u)/|u|^2 \to \infty$ as $|u| \to \infty$ uniformly on Ω, one can easily prove that Theorem 1.2 still holds. Then it will be applicable to functions that can't be of q-order growth, e.g. $G(u) = u^2 \ln(e^2 + u^2)$.

Remark 1.4. Theorem 1.1 of this paper improves and extends [10, Theorem 3.7] by replacing the (AR) condition (f_2) by a weaker form. For example let

$$G(u) = |u|^4 + 2|u|^{3.5} \sin^2\left(2|u|^{0.5}\right),$$

then it satisfies Theorem 1.2 for $N = 3$ but not verifying the (AR) condition. We also note that our Theorem 1.2 can be viewed as the complement of [7, Theorem 4] and [6, Theorem 1], which only obtained the existence of a nontrivial solution of (1.1) via the local linking theorem in [7]. In [5], the conditions (A_1)–(A_4) can imply our (S_1)–(S_3). This indicates that Theorem 1.2 in this paper is more general than [5, Theorem 1.1].