J. Math. Anal. Appl. 414 (2014) 296–318

Contents lists available at ScienceDirect

Journal of Mathematical Analysis and Applications

www.elsevier.com/locate/jmaa

Debt–equity swap with finite time horizon—variational inequality approach ☆

Xiaoru Han [a,b], Fahuai Yi [a,*], Jianbo Zhang [c]

[a] School of Mathematical Sciences, South China Normal University, Guangzhou, China
[b] Department of Mathematics, Foshan University, Guangdong 528000, China
[c] Department of Economics, University of Kansas, Lawrence, KS 66045, USA

ARTICLE INFO

Article history:
Received 26 September 2013
Available online 8 January 2014
Submitted by H.-M. Yin

Keywords:
Free boundary problem
Variational inequality
Mathematical finance
Equity valuation
Debt–equity swap

ABSTRACT

This paper concerns the finite-horizon optimal reorganization problem under debt–equity swap. The model of equity is formulated as a parabolic variational inequality, or equivalently, a free boundary problem, where the free boundary corresponds to the optimal reorganization boundary. The existence and uniqueness of the solution are proven and the behavior of the free boundary, such as smoothness, monotonicity and boundedness, is studied. To the best of our knowledge, this is the first complete set of results on debt–equity swap for finite maturity obtained using PDE techniques.

1. Introduction

Merton [12] pioneers the study of corporate debt and equity value under the assumption that the assets of the firm secure the debt, but the debt holders cannot force the firm to bankrupt before maturity T, he obtained the value of equity and debt.

Blank and Cox [2], Geske [7], Leland [9], Anderson and Sundaresan [1], Mella-Barral and Perraudin [11], Fan and Sundaresan [4] extended Merton's model to the more realistic assumption with the possibility of early default. Leland [9] introduced perpetual debt, tax and bankruptcy cost into the debt pricing model, and obtained the optimal capital structure. Fan and Sundaresan [4],under assumption that at an endogenously determined reorganization boundary debtholders are offered a proportion of the firm's equity to replace the original debt contract, given the equity holders' bargaining power parameter, η, obtained the optimal sharing rule between both equityholder and debtholder using Nash bargaining solution. They also characterized the infinite-horizon optimal trigger point for debt–equity swap and derived equity and debt value.

Although perpetual debt allows us to derive closed-form solution, it is still important to examine debt contracts with finite maturity. Finite maturity debt contracts are most commonly issued by firms and traded

☆ The project is supported by National Natural Science Foundation of China (No. 11271143 and 11371155) and University Special Research Fund for Ph.D. Program of China (20124407110001 and 20114407120008).
* Corresponding author.

0022-247X/$ – see front matter
http://dx.doi.org/10.1016/j.jmaa.2013.12.067

around the world. The essence of this important modification lies in that the debt values are no longer time homogeneous. The fundamental valuation equation will explicitly depend on time.

This paper concerns the finite-horizon optimal reorganization and aims to provide a theoretical analysis of the optimal reorganization policies under debt–equity swap. Under this scheme, a natural problem is how to determine the optimal reorganization boundary, in addition to valuating the equity.

One paper related to the present work is Sahakyan [14] where the author considered the debt with no coupon payments. He characterized the integral-differential equation for the reorganization boundary and provided an analytical solution for two special situations: when sharing rule provides debtholders with either all assets or nothing upon default. It should be noted that his approach cannot be extended to the general case with coupon payments or general sharing rules.

We will attack the problem directly using a variational inequality approach. We shall see the equity valuation problem in this scheme leads to a parabolic variational inequity, or equivalently, a free boundary problem, where the free boundary corresponds to the optimal reorganization boundary. Since the difference between the model (2.3) and the American option pricing model lies in that the obstacle function is different from the terminal condition, which causes that the inequality $\partial_\tau u \geqslant 0$ (τ is the time to maturity) not to hold in the case of $c(1-\gamma)-rP>0$. However, it is well known that the condition $\partial_\tau u \geqslant 0$ is critical in the proof of smoothness and monotonicity of the free boundary for the heat equation in [5]. At this point, it brings difficulty for analyzing the behavior of the free boundary.

The rest of this paper is arranged as follows. In the next section, we present the model formulation. In Section 3, we study the behavior of solution. In Section 4, using the similar method in [5] and [16], we prove that the free boundary $h(\tau)$ is decreasing and infinitely differentiable for the case, $c(1-\gamma)-rP \leqslant 0$. Our main contribution is in Section 5, where we analyze the behavior of the free boundary (i.e. optimal reorganization polices) for the case, $c(1-\gamma)-rP>0$. First, we make a variable transformation to obtain a similar property to $\partial_\tau u \geqslant 0$ and show the smoothness of the free boundary. Then, we prove the boundedness of the free boundary by comparison principle. Finally, we show that the free boundary is not always monotonic by providing a counter-example. In Section 6 we provide some numerical results. In the last section we present some financial interpretations of the mathematical results in this paper. The main contribution of this paper lies in the following.

1. Formal derivation of variational inequality (2.3) by stochastic analysis (Section 2).
2. Making a transformation to regain the infinite differentiability of the reorganization boundary for the case of $c(1-\gamma)-rP>0$ (Section 5).
3. Constructing comparison function (5.16) to obtain the lower bound of the reorganization boundary for the case of $c(1-\gamma)-rP>0$ (Section 5).
4. The loss of monotonicity of the reorganization boundary in some cases (Section 5).
5. Some numerical results and financial interpretations (Sections 6 and 7).

2. Formulation of the model

In this section, we develop a model of equity value under the debt–equity swap with finite time maturity at time T. The model is set in a continuous-time framework. The following assumptions underlie the model:

(1) There is a firm which has equity and a single issue of debt which promises a flow rate of coupon c per unit time. The principle amount of the debt is P.

(2) To focus attention on default risk, we assume that the default-free term structure is flat and the instantaneous risk-free rate is r per unit time.

(3) When the firm pays its contractual coupon c, it is entitled to a tax benefit of γc ($0 \leqslant \gamma \leqslant 1$). During the default period, the tax benefits are lost.

(4) Asset sales for dividend payments are prohibited.