J. Math. Anal. Appl. 414 (2014) 319–333

Contents lists available at ScienceDirect

Journal of Mathematical Analysis and Applications

www.elsevier.com/locate/jmaa

Interpolation of analytic functions of moderate growth in the unit disc and zeros of solutions of a linear differential equation

Igor Chyzhykov [a,*], Iryna Sheparovych [b]

[a] *Faculty of Mechanics and Mathematics, Ivan Franko National University of Lviv, Universytets'ka 1, 79000, Lviv, Ukraine*
[b] *Institute of Physics, Mathematics and Computer Science, Drohobych Ivan Franko State Pedagogical University, Stryis'ka 3, Drohobych, Ukraine*

ARTICLE INFO

Article history:
Received 26 August 2013
Available online 8 January 2014
Submitted by Steven G. Krantz

Keywords:
Interpolation
Unit disc
Analytic function
Oscillation
Differential equation

ABSTRACT

In 2002 A. Hartmann and X. Massaneda obtained necessary and sufficient conditions for interpolation sequences for classes of analytic functions in the unit disc such that $\log M(r,f) = O((1-r)^{-\rho})$, $0 < r < 1$, $\rho \in (0,+\infty)$, where $M(r,f) = \max\{|f(z)|\colon |z| = r\}$. Using another method, we give an explicit construction of an interpolating function in this result. As an application we describe minimal growth of the coefficient a such that the equation $f'' + a(z)f = 0$ possesses a solution with a prescribed sequence of zeros.

1. Introduction and results

1.1. Interpolation in the unit disc

Let (z_n) be a sequence of different complex numbers in the unit disc $\mathbb{D} = \{z\colon |z| < 1\}$, and let $\sigma(z,\zeta) = |\frac{z-\zeta}{1-\bar{z}\zeta}|$ denote the pseudohyperbolic distance in $\mathbb{D}$. Let $U(z,t) = \{\zeta \in \mathbb{C}\colon |\zeta - z| < t\}$. In the sequel, the symbol C stands for positive constants which depend on the parameters indicated, not necessarily the same at each occurrence. We say that the sequence (z_n) is *uniformly discrete* or *separated*, if $\inf_{j\neq k} \sigma(z_k, z_j) > 0$. L. Carleson [2,8] considered the problem of description of the so-called *universal interpolation sequences* or *interpolation sets* for the class H^∞ of bounded analytic functions in $\mathbb{D}$, i.e. those sequences (z_k) in $\mathbb{D}$ that $\forall (b_k) \in l^\infty$ there exists $f \in H^\infty$ with

$$f(z_k) = b_k. \qquad (1)$$

* Corresponding author.
E-mail addresses: chyzhykov@yahoo.com (I. Chyzhykov), isheparovych@ukr.net (I. Sheparovych).

0022-247X/$ – see front matter
http://dx.doi.org/10.1016/j.jmaa.2013.12.066

He proved that (z_k) is a universal interpolation sequence for H^∞ if and only if

$$\exists\delta > 0: \quad \prod_{j\neq k} \sigma(z_j, z_k) \geqslant \delta, \quad k \in \mathbb{N}. \tag{2}$$

For the similar problems in H^p see [8, Chap. 9].

For the Banach space A^{-n}, $n > 0$, of analytic functions such that $\|f\|_n^\infty = \sup_{z\in\mathbb{D}}(1-|z|)^n|f(z)| < \infty$, an interpolation set is defined by the condition that for every sequence (b_k) with $(b_k(1-|z_k|)^n) \in l^\infty$ there is a function $f \in A^{-n}$ satisfying (1). These sets were described by K. Seip in [20]. Namely, necessary and sufficient that (z_k) be an interpolation set for A^{-n} is that (z_n) be separated and $\mathcal{D}^+(Z) < n$ where

$$\mathcal{D}^+(Z) = \varlimsup_{r\uparrow 1} \sup_{z\in\mathbb{D}} \frac{\sum_{\frac{1}{2}<\sigma(z,z_j)<r} \ln \frac{1}{\sigma(z,z_j)}}{\ln \frac{1}{1-r}}. \tag{3}$$

We note that the condition (2) implies boundedness of the numerator in (3).

For an analytic function f in $\mathbb{D}$ we denote $M(r,f) = \max\{|f(z)|: |z| = r\}$, $r \in (0,1)$. Let $n_\zeta(t) = \sum_{|z_k-\zeta|\leqslant t} 1$ be the number of the members of the sequence (z_k) satisfying $|z_k - \zeta| \leqslant t$. We write

$$N_\zeta(r) = \int_0^r \frac{(n_\zeta(t)-1)^+}{t}\, dt.$$

The results mentioned above cannot be applied to analytic functions f such that $\ln\frac{1}{1-r} = o(\ln M(r,f))$ $(r\uparrow 1)$. In 1956 A.G. Naftalevich [18] described interpolation sequences for the Nevanlinna class. On the other hand, a description of interpolation sets in the class of analytic functions in the unit disc and of infinite order of the growth satisfying

$$\exists C > 0\ \forall r \in (0;1): \quad \ln\ln M(r,f) \leqslant C \ln\gamma\left(\frac{C}{1-r}\right),$$

where $\ln\gamma(t)$ is a convex function in $\ln t$ and $\ln t = o(\ln\gamma(t))$ $(t\to\infty)$, was found by B. Vynnyts'kyi and I. Sheparovych in 2001 [28].

Consider the class of analytic functions such that

$$\exists C > 0\ \forall r \in (0;1): \quad \ln M(r,f) \leqslant C\eta\left(\frac{C}{1-r}\right), \tag{4}$$

where $\eta: [1,+\infty) \to (0,+\infty)$ is an increasing convex function in $\ln t$ such that $\ln t = o(\eta(t))$ $(t\to\infty)$. In 2001 in the PhD thesis of the second author [23, Theorem 3.1] (see also [29]) it was proved that given a sequence (z_n) in $\mathbb{D}$, in order that for every (b_n) such that

$$\exists C > 0\ \forall n \in \mathbb{N}: \quad \log|b_n| \leqslant C\eta\left(\frac{C}{1-|z_n|}\right)$$

there exists an analytic function from the class (4) satisfying (1), it is necessary that

$$\exists\delta \in (0,1)\ \exists C > 0\ \forall n \in \mathbb{N}: \quad N_{z_n}\big(\delta(1-|z_n|)\big) \leqslant \eta\left(\frac{C}{1-|z_n|}\right). \tag{5}$$

In 2002 A. Hartmann and X. Massaneda [11] proved that condition (5) is actually necessary and sufficient for a class of growth functions η containing all power functions. They also described interpolation sequences