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On random frequent universality

A. Mouze [a,b,*], V. Munnier [b]

[a] Laboratoire Paul Painlevé, UMR8524, Cité Scientifique, 59655 Villeneuve d'Ascq cedex, France
[b] École Centrale de Lille, Cité Scientifique, CS20048, 59651 Villeneuve d'Ascq cedex, France

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ABSTRACT

We give a probabilistic construction of frequently hypercyclic or universal entire functions using Birkhoff's ergodic Theorem. We apply the same ideas to construct random frequently universal vectors for the polynomials of the weighted backward shift on the classical real or complex ℓ^p space.

1. Introduction

In 2004 Bayart and Grivaux introduced the notion of frequently hypercyclic operators and gave a frequent hypercyclicity criterion [1,2]. Let Y, Z be topological vector spaces. An operator $T : Y \to Y$ is called *frequently hypercyclic* if there exists a vector f such that for every non-empty open set $U \subset Y$ the set of integers n such that the sequence of iterates $T^n f$ belongs to U has positive lower density, i.e.

$$\liminf_{N\to+\infty} \frac{\#\{k \in \{0,1,\dots,N\};\ T^k f \in U\}}{N+1} > 0,$$

where as usual # denotes the cardinality of the corresponding set. Let us recall that T is said to be *hypercyclic* if there is some vector $y \in Y$ such that the set $\{T^n y;\ n \in \mathbb{N}\}$ is dense in Y and the element y is then also called hypercyclic. Hence, roughly speaking the notion of frequent hypercyclicity appraises how often the orbit of a hypercyclic vector visits a non-empty open set. For instance the hypercyclic differentiation operator on the space $H(\mathbb{C})$ of entire function [12] is additionally frequently hypercyclic [2]. The notion of hypercyclicity can be viewed as a particular case of universality [10], where the sequence of the iterates of only one operator (T^n) is replaced by a sequence of operators (T_n) with $T_n : Y \to Z$. Thus the sequence (T_n) is said to be *universal* if there is some vector $y \in Y$ such that the set $\{T_n y;\ n \in \mathbb{N}\}$ is dense in Z. A large class of examples of such phenomena is given by the universal series, i.e. whenever the sequence (T_n) is a sequence of partial sums. We refer the reader to [3]. Moreover the sequence of operators (T_n) will be called *frequently universal* if there exists a vector f such that for every non-empty open set $U \subset Z$ the set $\{n \in \mathbb{N};\ T_n f \in U\}$ has positive lower density.

We make mention of [7] for a strengthened version of the frequent hypercyclicity criterion as well as a frequent universality criterion and several examples. To go further in the subject of the dynamics of linear operators, we refer to the recent book [4].

Recently Nikula gave an interesting probabilistic construction of frequently hypercyclic functions for the differentiation operator on $H(\mathbb{C})$ [13]. More precisely, he proved that random entire functions $f(z) = \sum_{n\geqslant 0} X_n \frac{z^n}{n!}$ with independent and

* Corresponding author.
E-mail addresses: Augustin.Mouze@math.univ-lille1.fr (A. Mouze), Vincent.Munnier@ec-lille.fr (V. Munnier).

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identically distributed coefficients (X_n) are almost surely frequently hypercyclic provided that the distribution X of the random complex variables X_n is supported by the whole complex plane and satisfies the following decay condition

$$\text{for some } \beta > 0, \quad \limsup_{r\to+\infty} (\log r)^{1+\beta}\mathbb{P}\big(|X| \geqslant r\big) < +\infty. \tag{1}$$

In the present work, we deal with the probabilistic construction of frequently hypercyclic or universal objects in more general settings. Actually we do not need to use the above mentioned criteria [1,2,7].

The paper is organized as follows. In Section 2 using Birkhoff's ergodic Theorem (see [4, Theorem 5.3]) we give a direct proof of Nikula's result under weaker assumption. In Section 3, combining the above strategy with the well-known Bohr's result on the radius of power series on complex plane, we obtain the frequent universality of specific generalized universal Taylor series which are related to the differentiation operator (see Theorem 3.2 and [8] for the notion of generalized universal series). In particular we show that the generalized partial sums $\sum_{k=0}^{n} X_{n+k}\frac{z^k}{k!}$ are almost surely frequently universal provided that the sequence of the random complex variables X_n satisfies the same weak assumption as in Section 2. This result does not seem to be a straightforward application of a frequently universal criterion. Notice that Papachristodoulos, Kyrezi and Nestoridis have already begun the study of frequent universal series [14,11]. In Section 4, we keep the same approach to prove new results on polynomials of the backward shift on the real or complex ℓ^p space, endowed with its natural norm $\|\cdot\|_p$. Indeed, for $p \geqslant 1$, we consider B_w the weighted backward shift on ℓ^p defined by $B_w(a_0, a_1, a_2, \ldots) = (w_1 a_1, w_2 a_2, \ldots)$, where $(w_n)_{n\geqslant 0}$ is any bounded sequence of non-zero scalars. Let P be a polynomial. Let us also consider the polynomial of the backward shift $T_w := P(B_w)$. Recently Conejero and Martínez-Giménez obtained the chaoticity of $P(B)$ under conditions of the coefficients of P, where $B = B_{(1,1,1,\ldots)}$ is the backward shift operator [9]. In our case, under similar conditions on the size of the coefficients of P, we exhibit a sequence of polynomials (u_n) such that we have $T_w(\sum_{n\geqslant 0} X_n u_n) = \sum_{n\geqslant 0} X_{n+1}u_n$. Thus the polynomial of the weighted backward shift acts as a shift operator. So we show that the element $\sum_{k\geqslant 0} X_k u_k$ is almost surely frequently hypercyclic for the operator T_w, provided that the sequence of complex (or real) random variables (X_n) satisfies an unrestrictive decay condition. Similarly, on the space $E_p = \{\sum_{k\geqslant 0} x_k u_k;\ \sum_{k\geqslant 0} |x_k|\|u_k\|_p < +\infty\} \subset \ell^p$, we prove that the element $\sum_{k\geqslant 0} X_k u_k$ is almost surely frequently universal for the sequence of operators $(T_{w,n})$ defined by $T_{w,n} : E_p \to \ell^p$, $\sum_{k\geqslant 0} x_k u_k \mapsto \sum_{k=0}^{n} x_{k+n}u_k$, whenever the sequence of complex (or real) random variables (X_n) satisfies the condition (1). Finally some improvements are discussed.

Notations. Throughout the paper, $(\Omega, \mathcal{B}, \mathbb{P})$ will be a standard probability space. We will say that the support of a complex random variable X is the *whole complex plane* if for every non-empty open set $U \subset \mathbb{C}$, we have $\mathbb{P}(X^{-1}(U)) > 0$.

2. Random power series and frequent hypercyclicity

In this section, we give a slight improvement of [13, Theorem 1] as a straightforward application of Birkhoff's ergodic Theorem. First of all, we prove two lemmas.

Lemma 2.1. *Let X be a complex random variable and (X_n) a sequence of independent copies of X. Assume that the random variable X satisfies the following property*

$$\textit{for all } r > 0, \quad \sum_{n=0}^{+\infty} \mathbb{P}\left(|X| \geqslant \frac{n!}{r^n}\right) < +\infty. \tag{2}$$

Then almost surely the power series $\sum_{n\geqslant 0} X_n \frac{z^n}{n!}$ represents an entire function.

Proof. From (2), applying Borel–Cantelli Lemma, we get

$$\text{for all } r > 0, \quad \mathbb{P}\left(\bigcap_{n\geqslant 0}\bigcup_{k\geqslant n}\left\{|X_k| \geqslant \frac{k!}{r^k}\right\}\right) = 0.$$

Hence, for any $r > 0$, we almost surely have,

$$\limsup_{n\to+\infty} |X_n|\frac{r^n}{n!} \leqslant 1.$$

So almost surely the power series $\sum_{n\geqslant 0} X_n \frac{z^n}{n!}$ represents an entire function. □

Lemma 2.2. *Let X be a complex random variable such that the support of the distribution of X is the whole complex plane. Let (X_n) be a sequence of independent copies of X. Assume that the random variable X satisfies the property (2). Denote by f the random element*

$$f(z) = \sum_{n\geqslant 0} X_n \frac{z^n}{n!}.$$