J. Math. Anal. Appl. 412 (2014) 720–743

Contents lists available at ScienceDirect

Journal of Mathematical Analysis and Applications

www.elsevier.com/locate/jmaa

Ribaucour transformations for flat surfaces in the hyperbolic 3-space ☆

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ARTICLE INFO

Article history:
Received 11 April 2013
Available online 5 November 2013
Submitted by H.R. Parks

Keywords:
Ribaucour transformations
Flat surface
Hyperbolic space
Front waves

ABSTRACT

We consider Ribaucour transformations for flat surfaces in the hyperbolic 3-space, $\mathbb{H}^3$. We show that such transformations produce complete, embedded ends of horosphere type and curves of singularities which generically are cuspidal edges. Moreover, we prove that these ends and curves of singularities do not intersect. We apply Ribaucour transformations to rotational flat surfaces in $\mathbb{H}^3$ providing new families of explicitly given flat surfaces $\mathbb{H}^3$ which are determined by several parameters. For special choices of the parameters, we get surfaces that are periodic in one variable and surfaces with any even number or an infinite number of embedded ends of horosphere type.

1. Introduction

It is well known that a complete flat surface in the hyperbolic 3-space $\mathbb{H}^3$ must be a horosphere or a hyperbolic cylinder. Excluding these surfaces, flat surfaces in $\mathbb{H}^3$ have singularities. It is a classical fact that any such surface, away from umbilic points, is characterized by a harmonic function in terms of its first and second fundamental forms. Moreover, such surfaces, whose second fundamental form is locally related to the classical Monge–Ampère equation $Det(\nabla^2 f) = 1$, have a Weierstrass-type representation formula obtained by J.A. Gálvez, A. Martínez and F. Milán [9].

Examples of flat surfaces with an isolated singularity appear in the family of flat rotational surfaces in $\mathbb{H}^3$. J.A. Gálvez and P. Mira gave in [8] a complete description of the flat surfaces that are regularly embedded around an isolated singularity and in [6], A.V. Corro, A. Martínez and F. Milán provided examples of such surfaces with two isolated singularities. P. Roitman [17] studied the geometric properties of flat surfaces motivated by a classical result of L. Bianchi.

Recent papers on the Cauchy problem for certain Monge–Ampère type equations (see [1,2,8,16]), motivate the interest for flat surfaces with singularities. Flat fronts were introduced by M. Kokubu, M. Umehara and K. Yamada [10] as flat surfaces with "admissible" singularities. A flat front can be defined as a map f of 2-dimensional manifold M^2 into a 3-dimensional space form with a well defined unit normal field ν such that the following property holds: there exists a neighborhood U of each point $p \in M$ such that either f restricted to U is a flat immersion or p is a singular point and the parallel map at a distance $t \neq 0$, for all t sufficiently small, is a flat immersion restricted to U. The geometry of flat fronts in $\mathbb{H}^3$ has been studied in Kokubu et al. [10]. Moreover, in [11] and [12] investigated the singularities of such surfaces. However, there are very few flat surfaces in $\mathbb{H}^3$ explicitly known.

☆ Research partially supported by Ministerio de Educación Grants No. MTM2010-19821, No. PHB2010-0109, Junta de Anadalucía Grants No. FQM325, No. P06-FQM-01642, Ministério de Ciência e Tecnologia, CNPq Proc. No. 303774/2009-6, Ministério de Educação, CAPES/DGU Proc. No. 23038010833/2010-37.

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0022-247X/$ – see front matter
http://dx.doi.org/10.1016/j.jmaa.2013.10.080

In this paper, we investigate some properties of the Ribaucour transformation for flat surfaces in the hyperbolic space and we produce new families of such surfaces given by explicit parametrizations, by applying Ribaucour transformations to the rotational flat surfaces.

Ribaucour transformations were classically studied by Ribaucour and Bianchi [3] to obtain new surfaces of constant Gaussian curvature from a given such surface. Although Ribaucour transformations for minimal surfaces were also classically known, the first examples were given by A.V. Corro, W. Ferreira and K. Tenenblat [4]. They also obtained a generalization of Ribaucour transformations to linear Weingarten surfaces of $\mathbb{R}^3$ in [5], providing also a unified version of the classical results. K. Tenenblat and Q. Wang [19] extended such transformations to surfaces of space forms. These transformations were applied to produce families of complete linear Weingarten surfaces and cmc surfaces in space forms from a given such surface. M. Lemes, P. Roitman, K. Tenenblat and R. Tribuzy [13] proved that the only Ribaucour transformations for linear Weingarten surfaces which are conformal are those relating surfaces with the same constant mean curvature. In [4], it was shown that Ribaucour transformations for minimal surfaces in $\mathbb{R}^3$ produce embedded planar ends. Moreover, in [13], such transformations for constant mean curvature 1 in $\mathbb{H}^3$ were shown to produce embedded horosphere type ends.

In this paper, we will show that Ribaucour transformations for flat surfaces in $\mathbb{H}^3$ produce curves of singularities and complete embedded ends of horosphere type. We will apply the Ribaucour transformations to the rotational flat surfaces in $\mathbb{H}^3$ and obtain families of new such surfaces. Each family has a distinguished class of surfaces which are periodic in one variable. They can be generated with any even number $2n$ of complete ends of horosphere type and 2 complete ends with geometric index m, where n/m is any irreducible rational number. One can visualize some of the flat surfaces associated to the cylinder in Figs. 1–12. Surfaces obtained from the rotational flat surfaces without isolated singularities are given in Figs. 13–21 and some of the surfaces associated to the flat surfaces with an isolated singularity can be visualized in Figs. 22–25.

In Section 2, we recall basic facts of flat surfaces in the hyperbolic 3-space, and the Ribaucour transformations for such surfaces. Moreover, we prove that these transformations for flat surfaces in $\mathbb{H}^3$ produce embedded ends of horosphere type and curves of singularities which are generically cuspidal edges. Moreover, we prove that these ends and curves of singularities do not intersect. In Section 3, we describe all flat surfaces of the hyperbolic 3-space obtained by applying the Ribaucour transformation to the hyperbolic cylinder. The families of flat surfaces associated to the rotational flat surfaces without isolated singularities are treated in Section 4. In Section 5, we obtain the families of flat surfaces associated by a Ribaucour transformation to the rotational flat surfaces with an isolated singularity. In each section, we also determine ends of horosphere type and curves of singularities produced by the transformation. The family of new flat surfaces obtained in Section 3 depends on one parameter, while the families of new surfaces obtained in Sections 4 and 5 depend on several parameters. In each family of flat surfaces, we distinguish a class of surfaces which are periodic in one variable. These surfaces are obtained by appropriately choosing the parameters. In particular we can choose the surfaces with any even number of embedded ends of horosphere type.

2. Ribaucour transformation for flat surfaces in $\mathbb{H}^3$

In this section, we recall the main results that will be used in the following section. Namely, the fact that Ribaucour transformations provide an integrable system of differential equations whose solutions enable us to obtain flat surfaces in a hyperbolic 3-space, from a given such surface. For more details and proofs see [19] and its references.

Let $\mathbb{L}^4$ be the set of points $x = (x_0, x_1, \dots, x_3) \in \mathbb{R}^4$ endowed with the pseudo-Riemannian inner product given by $\langle x, y\rangle = -x_0y_0 + \sum_{i=1}^3 x_iy_i$. We consider the hyperbolic three space, with constant sectional curvature -1, as the submanifold of $\mathbb{L}^4$, $\mathbb{H}^3 = \{x \in \mathbb{L}^4 \mid \langle x, x\rangle = -1\}$, with two connected components. It will be useful in the following section. We consider M a flat surface in $\mathbb{H}^3$ admitting a parametrization $X(u_1, u_2)$ by lines of curvature. This is a regular parametrization except at umbilic points. Since umbilic points are isolated except for horospheres. These are the only surfaces excluded from consideration. In that situation, the second fundamental form is determined by the first fundamental form as given in the following well known result (see for example [18, p. 8 or p. 15]).

Proposition 2.1. *Away from umbilic points, a flat surface M in $\mathbb{H}^3$ admits a parametrization $X(u_1, u_2)$ by lines of curvature in such a way that*

$$I = \cosh^2\phi\, du_1^2 + \sinh^2\phi\, du_2^2,$$

where ϕ is a positive harmonic function. Moreover, the second fundamental form is given by

$$II = \pm\frac{1}{2}\sinh(2\phi)\big(du_1^2 + du_2^2\big).$$

The simplest examples of flat surfaces in $\mathbb{H}^3$ are characterized by the harmonic functions $\phi = b$, where $b \neq 0$ is a real number, $\phi = bu_2$, with $b \neq 0, 1$ or $\phi = bu_1$. These harmonic functions correspond, respectively, to the cylinder, the rotational flat surfaces without isolated singularities and the rotational flat surfaces with isolated singularities. We observe that when $\phi = u_2$ one has a flat surface which is not a rotational one called peach front (see [11,15]).