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On the set of points with absolutely convergent trigonometric series

Baowei Wang, Jun Wu *

School of Mathematics and Statistics, Huazhong University of Science and Technology, Wuhan, 430074, PR China

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ABSTRACT

Let $\{n_k\}_{k\geqslant 1}$ be a sequence of real numbers. Denote $E(\{n_k\}_{k\geqslant 1})$ the set of points for which the trigonometric series $\sum_{k\geqslant 1}\sin(n_k x)$ converges absolutely. It is shown that if $t_k := n_{k+1}/n_k$ tends to infinity monotonically, the Hausdorff dimension of $E(\{n_k\}_{k\geqslant 1})$ is given by the formula $1-\limsup_{k\to\infty}\frac{\log k}{\log t_k}$; if not, this dimensional formula may be false. This strengthens a former work of Erdös and Taylor.

1. Introduction

Let $\{n_k\}_{k\geqslant 1}$ be a sequence of real numbers. The distribution of the sequence $\{n_k x\}_{k\geqslant 1}$ is a fundamental topic in number theory (see the book [9] and the references therein). One of the most celebrated works on this topic is Weyl's criterion for uniform distribution [15] (see also [9]). From Weyl's criterion, we know that if there exists a constant $\delta > 0$ such that $|n_k - n_\ell| \geqslant \delta$ for all $k \neq \ell$, then for almost all $x \in \mathbb{R}$, the sequence $\{\langle n_k x\rangle\}_{k\geqslant 1}$ is uniformly distributed among the unit interval $[0, 1]$, where $\langle\cdot\rangle$ denotes the fractional part.

As a complement to this almost sure result, a natural question is to study the points violating this almost sure result. Erdös asked for the first time whether there exist points x such that $\{\langle n_k x\rangle\}_{k\geqslant 1}$ can be isolated from zero [5]. This was studied by Pollington [12] and de Mathan [11] who showed that if $\{n_k\}_{k\geqslant 1}$ is a lacunary sequence (that is, there exists $\lambda > 1$ such that $n_{k+1}/n_k \geqslant \lambda$ for all $k \geqslant 1$), the set of points for which $\{\langle n_k x\rangle\}_{k\geqslant 1}$ is isolated from zero is of full Hausdorff dimension. Moreover, Boshernitzan [1] showed that if $n_{k+1}/n_k \to 1$ as $k \to \infty$, the set of points for which $\{\langle n_k x\rangle\}_{k\geqslant 1}$ is non-dense in $[0, 1]$ is of Hausdorff dimension zero. As a further investigation on the points violating the almost sure result, Erdös and Taylor [6] also studied the points x for which $\langle n_k x\rangle$ converges to a given point in a fast speed. More precisely, they considered the dimension of the set

$$E(\{n_k\}_{k\geqslant 1}) = \left\{x \in [0, \pi]\colon \sum_{k\geqslant 1}\left|\sin(n_k x)\right| < \infty\right\}$$

i.e., the set of points for which the trigonometric series $\sum_{k\geqslant 1}\sin(n_k x)$ converges absolutely. Let $\{t_k = n_{k+1}/n_k\}_{k\geqslant 1}$ be the ratios of two consecutive terms of the sequence $\{n_k\}_{k\geqslant 1}$. In [6], they showed that

Theorem 1.1. *(See Erdös and Taylor [6].) (1) If $\{t_k\}_{k\geqslant 1}$ is bounded, $E(\{n_k\}_{k\geqslant 1})$ is at most countable.*

(2) If $\{t_k\}_{k\geqslant 1}$ fulfills the condition that

$$uk^\rho \leqslant t_k \leqslant vk^\rho, \quad \text{for all } k \geqslant 1 \tag{1}$$

* Corresponding author.
E-mail addresses: bwei_wang@hust.edu.cn (B. Wang), jun.wu@mail.hust.edu.cn (J. Wu).

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for some absolute constants $u, v, \rho > 0$, *the Hausdorff dimension of the set* $E(\{n_k\}_{k\geqslant 1})$ *is given by*

$$\dim_H E(\{n_k\}_{k\geqslant 1}) = 1 - \frac{1}{\max\{1, \rho\}}.$$

In this note, we intend to release the condition (1). Our first result is

Theorem 1.2. *If* $\{t_k\}_{k\geqslant 1}$ *tends to infinity monotonically, the dimension of* $E(\{n_k\}_{k\geqslant 1})$ *is given by the formula*

$$\dim_H E(\{n_k\}_{k\geqslant 1}) = 1 - \min\left\{\limsup_{k\to\infty} \frac{\log k}{\log t_k}, 1\right\} = 1 - \min\{\alpha_0, 1\} \tag{2}$$

where α_0 *is the convergence exponent of the sequence* $\{t_k\}_{k\geqslant 1}$, *i.e.*

$$\alpha_0 = \inf\left\{\alpha \geqslant 0: \sum_{k\geqslant 1} \frac{1}{t_k^\alpha} < \infty\right\}. \tag{3}$$

At the same time, we also provide a counter-example indicating that if $\{t_k\}_{k\geqslant 1}$ does not tend to infinity monotonically, the dimensional formula given in (2) may be false.

Theorem 1.3. *There exists a sequence of real numbers* $\{n_k\}_{k\geqslant 1}$ *such that*

$$\limsup_{k\to\infty} \frac{\log k}{\log t_k} = 1, \qquad \alpha_0 = 1, \quad \textit{but } \dim_H E(\{n_k\}_{k\geqslant 1}) = 1.$$

Remark 1.4. With no more difficulties, by the same method used for proving Theorem 1.2, we can show that the same result holds for the set

$$E(\{n_k, y_k\}_{k\geqslant 1}) = \left\{x \in \mathbb{R}: \sum_{k\geqslant 1} \left|\sin(n_k x - y_k)\right| < \infty\right\}$$

where $\{y_k\}_{k\geqslant 1}$ is any given sequence of reals.

As said before, the set $E(\{n_k\}_{k\geqslant 1})$ consists of the points x for which $\langle n_k x\rangle$ tends to zero with a fast speed. In this case, to avoid the possibility that the set $E(\{n_k\}_{k\geqslant 1})$ may be empty, the sequence $\{n_k\}_{k\geqslant 1}$ should tend to infinity in a very fast manner. If one's concern is only the convergence of $\langle n_k x\rangle$, things will be much different. Eggleston [4] showed that once $n_{k+1}/n_k \to \infty$, the set of points x for which $\langle n_k x\rangle$ tends to zero is of full Hausdorff dimension. While Bugeaud [2] (see also [3]) showed that, even if n_k tends to infinity very slowly there can always exist points x for which $\langle n_k x\rangle$ tends to a given point. More precisely, for any $\xi \in [0, 1]$ and any g_k tending to infinity as $k \to \infty$, there exists a sequence $\{n_k\}_{k\geqslant 1}$ with $n_k \leqslant k g_k$ such that the set

$$\left\{x \in [0, 1]: \lim_{k\to\infty} \langle n_k x\rangle = \xi\right\}$$

is non-empty. See also [8,10,14] for the convergence of $\langle n_k x\rangle$ when $\{n_k\}_{k\geqslant 1}$ is the sequence of the denominators of the convergents in the continued fraction expansion of some point.

2. Proof of Theorem 1.2: Lower bound

The following lemma indicates the relationship between the growth speed of $\{t_k\}_{k\geqslant 1}$ and its convergence exponent when $\{t_k\}_{k\geqslant 1}$ is monotone.

Lemma 2.1. *(See [13, p. 26].) If* $\{t_k\}_{k\geqslant 1}$ *is monotonic, its convergence exponent* α_0 *and* $\limsup_{k\to\infty} \frac{\log k}{\log t_k}$ *coincide, i.e.*

$$\alpha_0 = \limsup_{k\to\infty} \frac{\log k}{\log t_k}. \tag{4}$$

From this lemma, the second equality in Theorem 1.2 is trivial.

We begin with a classic lemma concerning the lower bound of the Hausdorff dimension of a set [7, Example 4.6].