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Partial regularity for sub-elliptic systems with Dini continuous coefficients involving natural growth terms in the Heisenberg group ☆

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ABSTRACT

We consider nonlinear sub-elliptic systems with Dini continuous coefficients in divergence form in the Heisenberg group. Based on a generalization of the method of $\mathcal{A}$-harmonic approximation introduced by Duzaar and Steffen, partial regularity of weak solutions for sub-elliptic systems under natural growth conditions is established. In particular, our result is optimal in the sense that in the case of Hölder continuous coefficients we obtain directly the optimal Hölder exponent for the horizontal gradients of weak solutions on the regular set.

1. Introduction and statements of main results

In this paper, we are concerned with partial regularity for nonlinear sub-elliptic systems involving quadratic natural growth term in the Heisenberg group $\mathbb{H}^n$ in divergence form, and more precisely, we consider the following systems

$$-\sum_{i=1}^{2n} X_i A_i^\alpha\big(\xi, u(\xi), Xu(\xi)\big) = B^\alpha\big(\xi, u(\xi), Xu(\xi)\big) \quad \text{in } \Omega, \tag{1.1}$$

where Ω is a bounded domain in $\mathbb{H}^n$, $\alpha = 1, 2, \dots, N$, $X = \{X_1, \dots, X_{2n}\}$, and the definition of X_i $(i = 1, \dots, 2n)$ is given in the next section in Eq. (2.2), $u = (u^1, \dots, u^N) : \Omega \to \mathbb{R}^N$, $A_i^\alpha(\xi, u, p) : \mathbb{R}^{2n+1} \times \mathbb{R}^N \times \mathbb{R}^{2nN} \to \mathbb{R}^{2nN}$, and $B^\alpha(\xi, u, p) : \mathbb{R}^{2n+1} \times \mathbb{R}^N \times \mathbb{R}^{2nN} \to \mathbb{R}^N$.

Under the coefficients A_i^α assumed to be Dini continuous, the aim of this paper is to establish optimal partial regularity to the sub-elliptic system (1.1) in the Heisenberg group $\mathbb{H}^n$. Comparing Hölder continuous coefficients; see [28,27] for the case of sub-elliptic systems, such assumption is more weaker. More precisely, we assume for the continuity of A_i^α with respect to the variables (ξ, u) that

$$\big(1 + |p|\big)^{-1}\big|A_i^\alpha(\xi, u, p) - A_i^\alpha(\tilde{\xi}, \tilde{u}, p)\big| \leqslant \kappa\big(|u|\big)\mu\big(d(\xi, \tilde{\xi}) + |u - \tilde{u}|\big) \tag{1.2}$$

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for all $\xi, \tilde{\xi} \in \Omega$, $u, \tilde{u} \in \mathbb{R}^N$, and $p \in \mathbb{R}^{2nN}$, where $\kappa : (0, +\infty) \to [1, +\infty)$ is nondecreasing, and $\mu : (0, +\infty) \to [0, +\infty)$ is nondecreasing and concave with $\mu(0+) = 0$. We also has required that $r \to r^{-\gamma}\mu(r)$ is nonincreasing for some $\gamma \in (0, 1)$ and that

$$M(r) = \int_0^r \frac{\mu(\rho)}{\rho}\, d\rho < \infty \quad \text{for some } r > 0. \tag{1.3}$$

We adopt the method of $\mathcal{A}$-harmonic approximation to the case of sub-elliptic systems in the Heisenberg groups, and establish the optimal partial regularity result: Roughly speaking, assume additionally to the standard hypotheses (see precisely (H1), (H2) and (H4) below) that $(1 + |p|)^{-1} A_i^\alpha(\xi, u, p)$ satisfies (1.2) and (1.3). Let $u \in HW^{1,2}(\Omega, \mathbb{R}^N)$ be a weak solution of the sub-elliptic system (1.1). Then u is of class C^1 outside a closed singular set $\operatorname{Sing} u \subset \Omega$ of Haar measure 0. Furthermore, for $\xi_0 \in \Omega \setminus \operatorname{Sing} u$, the derivative Xu of u has the modulus of continuity $r \to M(r)$ in a neighborhood of ξ_0. Our result is optimal in the sense that in the case $\mu(r) = r^\gamma$, $0 < \gamma < 1$, we have $M(r) = \gamma^{-1} r^\gamma$ Hölder continuity $\Gamma^{1,\gamma}$ to be optimal in that case.

As is well known, even under reasonable assumptions on A_i^α and B^α of the systems of equations, one cannot, in general, expect that weak solutions of (1.1) will be classical, i.e. C^2-solutions. This was first shown by De Giorgi [6]; we also refer the reader to Giaquinta [17], and Chen and Wu [5] for further discussion and additional examples. Then the goal is to establish partial regularity theory. Moreover, a new method called $\mathcal{A}$-harmonic approximation technique is introduced by Duzaar and Steffen in [13], and simplified by Duzaar and Grotowski in [10], to study elliptic systems with quadratic growth case. Then, similar results have been proved for more general A_i^α or B^α in the Euclidean setting; see [15,11,12,4] for Hölder continuous coefficients, and [9,14,24,25] for Dini continuous coefficients.

However, turning to sub-elliptic equations and systems in the Heisenberg groups $\mathbb{H}^n$, some new difficulties will arise due to non-commutativity of the horizontal vectors X_i; see (2.3). We refer readers to [7,8] by Domokos for more details. Several results were focused on those equations corresponding to basic vector fields on the Heisenberg group or more generally, Carnot group. Capogna [1,2] studied the regularities for weak solutions to quasi-linear equations, and established differentiability in the non-horizontal direction, $W^{2,2}$ estimate and C^∞ continuity; see [1] for the case of Heisenberg groups, and [2] for Carnot groups. To sub-elliptic p-Laplace equations in Heisenberg groups, Marchi in [20,22,21] showed that $Tu \in L^p_{\mathrm{loc}}$ and $X^2 u \in L^2_{\mathrm{loc}}$ for $1 + \frac{1}{\sqrt{5}} < p < 1 + \sqrt{5}$ using theories of Besov space and Bessel potential space. Domokos in [7,8] improved these results for $1 < p < 4$ employing Zygmund theory related to vector fields. Later, Manfredi and Mingione in [19] and Mingione, Zatorska-Goldstein and Zhong in [23] proved Hölder regularity with regard to full Euclidean gradient for weak solutions and further C^∞ continuity under the coefficients assumed to be smooth.

While regularities of weak solutions to sub-elliptic systems concerning vector fields are more complicated. Capogna and Garofalo in [3] showed partial regularity of weak solutions to quasi-linear sub-elliptic systems under quadratic natural growth conditions in Carnot groups of step two. Their way relies mainly on the generalization of the classical direct method in the Euclidean setting. Shores in [26] considered a homogeneous quasi-linear system under the quadratic growth condition in Carnot groups with general step. She established higher differentiability and smoothness for weak solutions of the system with constant coefficients, and deduced partial regularity for weak solutions to the homogeneous quasi-linear system. With respect to the case of non-quadratic growth, Föglein in [16] treated a homogeneous nonlinear system in the Heisenberg group under super-quadratic growth conditions. She got a priori estimate for a system with constant coefficients and proved partial regularity of weak solutions to the homogeneous nonlinear system. Recently, Wang and Niu [28], Wang and Liao [27] treated more general nonlinear sub-elliptic system in the Carnot groups under super-quadratic growth conditions, and sub-quadratic growth conditions, respectively.

The regularity results for sub-elliptic systems mentioned above require Hölder continuity with respect to the coefficients A_i^α. When the assumption of Hölder continuity on A_i^α is weakened to Dini continuity, how to establish partial regularity of weak solutions to nonlinear sub-elliptic systems under natural growth conditions in the Heisenberg group. This paper is devoted to this topic. To define weak solution to (1.1), we assume the following structure conditions on A_i^α and B^α:

(H1) $A_i^\alpha(\xi, u, p)$ is differentiable in p, and there exists some constant L such that

$$\left|A^\alpha_{i,p^j_\beta}(\xi, u, p)\right| \leqslant L, \quad (\xi, u, p) \in \Omega \times \mathbb{R}^N \times \mathbb{R}^{2nN}, \tag{1.4}$$

where we write down $A^\alpha_{i,p^j_\beta}(\xi, u, p) = \frac{\partial A_i^\alpha(\xi,u,p)}{\partial p^j_\beta}$.

(H2) $A_i^\alpha(\xi, u, p)$ is uniformly elliptic, i.e. for some $\lambda > 0$ we have

$$A^\alpha_{i,p^j_\beta}(\xi, u, p)\eta_i^\alpha \eta_j^\beta \geqslant \lambda|\eta|^2, \quad \forall \eta \in \mathbb{R}^{2nN}. \tag{1.5}$$

(H3) There exists a modulus of continuity $\mu : (0, +\infty) \to [0, +\infty)$, and a nondecreasing function $\kappa : [0, +\infty) \to [1, +\infty)$ such that

$$\big(1 + |p|\big)^{-1}\left|A_i^\alpha(\xi, u, p) - A_i^\alpha(\tilde{\xi}, \tilde{u}, p)\right| \leqslant \kappa\big(|u|\big)\mu\big(d(\xi, \tilde{\xi}) + |u - \tilde{u}|\big). \tag{1.6}$$