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A discussion on the coexistence of heteroclinic orbit and saddle foci for third-order systems

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ABSTRACT

The coexistence of heteroclinic orbits and saddle foci is concerned with the basic assumption in Shil'nikov heteroclinic theorem. Two aspects of this discussion are conducted in the paper. Firstly, many third-order systems, which possess exact heteroclinic orbits expressed by pure hyperbolic functions or the combination of hyperbolic and triangle functions and so on, have been constructed. At the same time, the existence of saddle foci is tested and some problems are proposed. Secondly and more importantly, the existence of heteroclinic orbits to saddle foci is studied. The necessary condition for the coexistence of heteroclinic orbits and saddle foci is obtained. Finally, an example is given to show the effectiveness of the results, and some conclusions and problems are presented.

1. Introduction

It has been noted that one of the commonly agreeable analysis of chaotic dynamics for third-order continuous-time autonomous systems is based on the fundamental work of Shil'nikov and its subsequent embellishments and slight extension [8–12]. This is known as the Shil'nikov heteroclinic theorem [12] or Shil'nikov criterion today, and its role is in some sense equivalent to that of Li–Yorke lemma in the discrete setting.

Consider the third-order autonomous system

$$\dot{x}(t) = f\big(x(t)\big), \quad t \in R,\ x(t) \in R^3 \tag{1.1}$$

where the vector field $f : R^3 \to R^3$, belongs to class c^r $(r \geqslant 2)$.

Let $x_e \in R^3$ be the equilibrium of system (1.1) and $Ax_e = DF|_{x_e}$ be the Jacobi matrix of system (1.1) at x_e.

Definition 1.1. x_e is called a saddle focus point if Ax_e possesses three eigenvalues with the following form

$$\nu, \quad \sigma \pm iw$$

where ν, σ and w all are real numbers and

$$\sigma\nu < 0, \quad \omega \neq 0. \tag{1.2}$$

The Shil'nikov heteroclinic theorem for the existence of chaos is the following lemma (referred to [12]).

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Lemma 1.2. *Suppose that two distinct equilibrium points, denoted by $x_e^{(1)}$ and $x_e^{(2)}$ respectively, of system (1.1) are saddle foci, whose characteristic values ν_k and $\sigma_k \pm i w_k$ $(k = 1, 2)$ satisfy the following Shil'nikov inequality*:

$$|\nu_k| > |\sigma_k| > 0, \quad k = 1, 2 \tag{1.3}$$

under constraint

$$\sigma_1\sigma_2 > 0 \quad or \quad \nu_1\nu_2 > 0. \tag{1.4}$$

Suppose also that there exists a heteroclinic loop joining $x_e^{(1)}$ to $x_e^{(2)}$. Then:

1. *The Shil'nikov map, defined in a neighborhood of the heteroclinic orbit, has a countable number of Smale horseshoes in its discrete dynamics*;
2. *For any sufficiently small C^1 perturbation g of f, the perturbed system*
$$\dot{x}(t) = g\big(x(t)\big), \quad x(t) \in R^3 \tag{1.5}$$
has at least a finite number of Smale horseshoes in the discrete dynamics of Shil'nikov map defined near the heteroclinic loop;
3. *Both the original system (1.1) and perturbed system (1.2) have horseshoe type of chaos.*

One can see that the existence of heteroclinic loop, which is made up of two heteroclinic orbits joining saddle focus one to anther, is the basic assumption of Lemma 1.2. The objective of this paper is to study the coexistence of heteroclinic orbit and saddle foci for third-order systems. It should be noted that the absence of heteroclinic orbit to saddle foci implies that there is no heteroclinic loop joining saddle foci.

The existence of the heteroclinic orbit [13,15,16], then heteroclinic loop, has been widely studied all the time. To my knowledge, there is no contribution concerned with exact heteroclinic orbit to saddle foci. Usually the authors of contributions adopt series approximation for heteroclinic orbit based on stable and unstable manifolds [5–7,13–16]. But the key problem of such method is that it needs that the system possesses large scale structure stability and C^1 close property. But it is almost impossible for general nonlinear system so that the convergence of approximate series cannot be guaranteed [1]. So, giving the exact expression of heteroclinic orbits [2,4] is important in the study.

Based on above analysis the author attempts to study the coexistence of heteroclinic orbits and saddle foci from two aspects. Firstly, a frame of constructing third-order systems with exact heteroclinic orbit is obtained. Under such frame we have constructed many third-order systems which possess exact heteroclinic orbit expressed by pure hyperbolic functions or the combination of hyperbolic and triangle functions and so on. At the same time, the existence of saddle foci is tested and some problems are proposed. Secondly, the existence of heteroclinic orbit to saddle foci is studied. The necessary condition for the coexistence of heteroclinic orbit and saddle foci is obtained.

The paper is organized as follows. In Section 2, we introduce a frame of constructing a third-order system with exact heteroclinic orbit, and the relative properties of such frame are also discussed. Based on such frame we construct many third-order systems which possess exact heteroclinic orbit expressed by pure hyperbolic functions or the combination of hyperbolic and triangle functions and so on. At the same time the existence of saddle foci is tested. It is discovered that corresponding eigenvalues of these systems with exact heteroclinic orbit cannot be satisfied with Definition 1.1. So, some problems are proposed. To solve some of these problems, as the main results of this paper, the necessary conditions for the system to be satisfied with the coexistence of heteroclinic orbit and saddle foci are obtained in Section 3. Finally, in Section 4 an example is given to applying the results and some conclusions are presented and some problems are proposed.

2. Constructing system $S(Y, H)$ with exact heteroclinic orbits and proposing some problems

2.1. A frame of constructing a third-order system with exact heteroclinic orbits

Let $Y = (y_1(t), y_2(t), \dots, y_m(t))^T \in \mathbb{R}^m$ $(m \geqslant 2)$ for $t \in (-\infty, +\infty)$ be a vector function with convergence property, i.e. there exist two constant vectors $e_1 = (a_1, a_2, \dots, a_m)$ and $e_2 = (b_1, b_2, \dots, b_m)$ such that $y_i(t) \to a_i$ $(i = 1, 2, \dots, m)$ as $t \to +\infty$ and $y_i(t) \to b_i$ $(i = 1, 2, \dots, m)$ as $t \to -\infty$.

In order to express more clearly, we introduce some concepts in the following.

Definition 2.1. $Y = (y_1(t), y_2(t), \dots, y_m(t))^T$ is said self-expressed with respect to its derivative $\dot{Y}(t)$ iff there exist continuous differentiable functions $g_i : \mathbb{R}^m \to \mathbb{R}^1$, $i = 1, 2, \dots, m$ such that $\dot{y}_i(t) = g_i(y_1(t), \dots, y_m(t))$, $i = 1, 2, \dots, m$.

In the sequel we call Y is self-expressed if it satisfies Definition 2.1 for simplicity. Let $x_i = h_i(y_1, \dots, y_m)$ $(i = 1, 2, 3)$ where $h_i : \mathbb{R}^3 \to \mathbb{R}^1$ $(i = 1, 2, 3)$ are continuous differentiable with respect to y_j $(j = 1, 2, 3)$.

The transformation H: $x_i = h_i(y_1, \dots, y_m)$ $(i = 1, 2, 3)$ is called invertible iff there exist functions $k_i : \mathbb{R}^3 \to \mathbb{R}^1$ $(i = 1, \dots, m)$ such that $y_i = k_i(x_1, x_2, x_3)$ $(i = 1, \dots, m)$ and $x_i = h_i(k_1(x_1, x_2, x_3), \dots, k_m(x_1, x_2, x_3))$ $(i = 1, 2, 3)$.