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Existence and non-existence results for nonlinear elliptic equations with nonstandard growth

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ABSTRACT

We consider the existence and non-existence of solutions for nonlinear elliptic equations whose model is

$$\begin{cases} -\operatorname{div}\left(|\nabla v|^{p(x)-2}\nabla v\right)+|v|^{q(x)-1}v=\mu & \text{in } \Omega, \\ v=0 & \text{on } \partial\Omega, \end{cases}$$

where Ω is a smooth bounded domain in $\mathbb{R}^N$ ($N \geqslant 2$), μ is a bounded Radon measure. Our results are twofold: when μ is absolutely continuous with respect to the $p(\cdot)$-capacity, the problem admits a unique solution; when μ is concentrated on a set of zero $r(\cdot)$-capacity ($r(\cdot) > p(\cdot)$ in $\overline{\Omega}$), then the problem does not admit a solution if $q(\cdot)$ is large enough.

1. Introduction

The subject of differential equations and spaces with variable exponents has undergone an explosive development in the past decades, see for example [12,13,17,19,20,23,26,35] and the references listed below. The flourishing growth is mainly motivated by their applications, which concerns flow through porous media [2], thermorheological fluids [1], image processing [9], and especially electrorheological fluids (an essential class of non-Newtonian fluids), which have not only been used in fast acting hydraulic valves and clutches, brakes, shock absorbers, but also in some new fields such as accurate abrasive polishing, robotics and space technology [28,29].

In this work, we consider the following nonlinear elliptic equations with variable exponents

$$\begin{cases} -\operatorname{div}(a(x,\nabla v))+|v|^{q(x)-1}v=\mu & \text{in } \Omega, \\ v=0 & \text{on } \partial\Omega, \end{cases} \tag{1.1}$$

where Ω is a smooth bounded domain in $\mathbb{R}^N$ ($N \geqslant 2$), μ is a bounded Radon measure and $p(\cdot):\overline{\Omega}\to(1,\infty)$ is log-Hölder continuous, i.e., there exists a positive constant C such that

$$-\left|p(x)-p(y)\right|\ln|x-y| \leqslant C, \quad \text{for every } x, y \in \overline{\Omega}. \tag{1.2}$$

We assume that $a:\Omega\times\mathbb{R}^N\to\mathbb{R}^N$ is a Carathéodory function (i.e. $a(x,\xi)$ is measurable on $\Omega, \forall\xi\in\mathbb{R}^N$, and $a(x,\xi)$ is continuous on $\mathbb{R}^N$ for a.e. $x\in\Omega$) such that

$$a(x,\xi)\xi \geqslant \alpha|\xi|^{p(x)}, \tag{1.3}$$

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$$\left|a(x,\xi)\right| \leqslant \beta\left[b(x)+|\xi|^{p(x)-1}\right], \tag{1.4}$$

$$\left(a(x,\xi)-a(x,\eta)\right)(\xi-\eta)>0, \tag{1.5}$$

for almost every $x \in \Omega$ and for all $\xi, \eta \in \mathbb{R}^N$ with $\xi \neq \eta$, where α, β are positive constants and $b(x)$ is a nonnegative function in $L^{p(\cdot)/p(\cdot)-1}(\Omega)$. The above assumptions (1.3)–(1.5) are natural generalizations of the classical Leray–Lions assumptions [24].

When $a(x,\xi)=|\xi|^{p(x)-2}\xi$, problem (1.1) reduces to the $p(x)$-Laplace equations, one of the most typical models with variable exponents, which have been studied widely in the past years, see for example [14–16,18,21,32] Recently, $p(x)$-Laplace equations with singular data, such as L^1 or measure data, have attracted much attention [3,4,25,30,31,33,34] In [25], the authors proved the existence of a suitable harmonic function satisfying a $p(x)$-type equations with positive measure data. In [30], the existence and uniqueness of entropy solutions were obtained for the elliptic $p(x)$-Laplace equations with L^1 data, while in [3] and [31], the existence and uniqueness results were also derived for the problem in the framework of renormalized solutions. These two definitions of solutions were then proved to be equivalent in [33]. In the parabolic case, the corresponding existence and uniqueness results were also obtained [4,34].

Although existence and uniqueness results for the $p(x)$-Laplace equations with L^1 data have been studied largely, as far as we know, few results have been obtained for the equations with general measure data. In the case where the external force term is a measure, the existence and non-existence mechanisms of $p(x)$-Laplace equations could be much more subtle [6–8,27]. Thus we consider in this paper the existence and non-existence results for problem (1.1) with general measure data.

We prove an existence and a non-existence results for problem (1.1) according to the singularity of data μ. Precisely speaking, when μ is absolutely continuous with respect to the $p(\cdot)$-capacity (i.e., μ does not charge sets of zero $p(\cdot)$-capacity), we prove that problem (1.1) admits a unique entropy solution. On the other hand, when μ is concentrated on a set of zero $r(\cdot)$-capacity ($r(\cdot)>p(\cdot)$ in $\overline{\Omega}$), then we prove that there does not exist an entropy solution to the problem for large $q(\cdot)$. Such a result implies that sets of zero $r(\cdot)$-capacity are in some sense removable singularities for the problem.

Now, let us end this section with some remarks on the organization of the paper. In Section 2, we provide some preliminaries about the function spaces and then we prove a useful decomposition lemma for "smooth" measures. In Section 3, we provide the existence result for problem (1.1), and then in Section 4, the last section, we prove a non-existence result for the problem.

2. Some preliminaries

2.1. Function spaces

The following basic theory about the variable exponent Lebesgue and Sobolev spaces can be found in [17,23], and the more recent monograph [13]. Set $1<p^-=\min_{x\in\overline{\Omega}} p(x)<p^+=\max_{x\in\overline{\Omega}} p(x)$. The variable exponent Lebesgue space can be defined as

$$L^{p(\cdot)}(\Omega)=\left\{u\colon\ \Omega\to\mathbb{R};\ u \text{ is measurable and } \int_\Omega |u|^{p(x)}\,dx<\infty\right\}$$

endowed with the Luxemburg norm

$$\|u\|_{L^{p(\cdot)}(\Omega)}=\inf\left\{\lambda>0\colon \int_\Omega \left|\frac{u(x)}{\lambda}\right|^{p(x)} dx\leqslant 1\right\}.$$

We have

$$\min\left\{\|u\|^{p^+}_{L^{p(\cdot)}(\Omega)}, \|u\|^{p^-}_{L^{p(\cdot)}(\Omega)}\right\}\leqslant \int_\Omega |u|^{p(x)}\,dx\leqslant \max\left\{\|u\|^{p^+}_{L^{p(\cdot)}(\Omega)}, \|u\|^{p^-}_{L^{p(\cdot)}(\Omega)}\right\}.$$

And moreover, let $r_i\in C(\overline{\Omega})$ with $r_i^->1$, $i=1,2$. Then if $r_1(x)\leqslant r_2(x)$ for any $x\in\overline{\Omega}$, the imbedding $L^{r_2(\cdot)}(\Omega)\hookrightarrow L^{r_1(\cdot)}(\Omega)$ is continuous, of which the norm does not exceed $|\Omega|+1$. As $p^->1$, the space is a reflexive Banach space with dual $L^{p'(\cdot)}(\Omega)$, where $\frac{1}{p(x)}+\frac{1}{p'(x)}=1$. Besides, for any $v\in L^{p'(\cdot)}(\Omega)$, we have Hölder's inequality

$$\int_\Omega |uv|\,dx\leqslant\left(\frac{1}{p^-}+\frac{1}{(p^-)'}\right)\|u\|_{L^{p(\cdot)}(\Omega)}\|v\|_{L^{p'(\cdot)}(\Omega)}.$$

For positive integer k, the generalized Sobolev space is defined as

$$W^{k,p(\cdot)}(\Omega)=\left\{u\in L^{p(\cdot)}(\Omega)\colon\ D^\alpha u\in L^{p(\cdot)}(\Omega),\ |\alpha|\leqslant k\right\}$$