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The roles of diffusivity and curvature in patterns on surfaces of revolution

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ABSTRACT

We address the question of finding sufficient conditions for existence as well as nonexistence of nonconstant stable stationary solution to the diffusion equation $u_t = \operatorname{div}(a\nabla u) + f(u)$ on a surface of revolution with and without boundary. Conditions found relate the diffusivity function a and the geometry of the surface where diffusion takes place. In the case where f is a bistable function, necessary conditions for the development of inner transition layers are given.

1. Introduction

The main concern in this paper is to find sufficient conditions for existence as well as nonexistence of nonconstant stable stationary solutions (herein referred to as *patterns*, for short) to the diffusion problem

$$u_t = \operatorname{div}_g\big(a(x)\nabla_g u\big) + f(u), \quad (t,x) \in \mathbb{R}^+ \times \mathcal{M} \tag{1.1}$$

where $\mathcal{M} \subset \mathbb{R}^3$ is a surface of revolution without boundary with metric g. The case where $\mathcal{M}$ has boundary will also be treated. The function a is smooth and positive and f is a function in $C^1(\mathbb{R})$, sometimes considered of bistable type.

This kind of problem appears as a mathematical model in many distinct areas and, roughly speaking, a solution models the time evolution of the concentration of a diffusing substance in a heterogeneous medium whose diffusivity is given by a positive function a, under the effect of a source or sink term f. Usually the diffusivity is a property of the material which the surface is made of.

Our concern herein is to find mechanisms of interaction between the diffusivity function a and the geometry of the domain so as to produce patterns to the problem (1.1) as well as those which do not produce patterns.

There is a vast literature addressing the question of nonexistence as well as existence of patterns to (1.1) in bounded domains of $\mathbb{R}^n$ when diffusivity is constant. It seems to have been first addressed in [3] and [15] for problems with Neumann boundary condition where it was proved that, for the case of constant diffusivity, no pattern exists if the domain is convex. If a is a constant function, nonexistence of patterns to (1.1) on a Riemannian manifold without boundary with nonnegative Ricci curvature was proved in [7], thus generalizing a similar result for surface of revolution found in [17]. In particular, if $\mathcal{M}$ is a surface of revolution the authors in [1] show that there are no patterns if the sum of the Gaussian curvature in every point p and the square of the geodesic curvature of the parallel passing through p is nonnegative.

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For bounded domains in $\mathbb{R}^N$ the question of how the diffusivity function can give rise to patterns, or not, has been considered by some authors.

For one-dimensional domains, i.e., when $\mathcal{M}$ is an interval, subjected to zero Neumann boundary condition, a sufficient condition for nonexistence of patterns was found to be $a'' < 0$ in [4] and $(\sqrt{a})'' < 0$ in [19]. In domains with dimension $N \geqslant 2$ this remains an open problem.

Still regarding one-dimensional domains, existence of a diffusivity function a which gives rise to patterns to (1.1) was addressed in [10,12]. These results were generalized to two-dimensional domains in [5] and for any dimension in [6].

Let us briefly mention our main results. To this end consider a smooth curve C in $\mathbb{R}^3$ parametrized by $x = (x_1, x_2, x_3) = (\psi(s), 0, \chi(s))$, $s \in [0, l]$ with $\psi(0) = \psi(l) = 0$ and the borderless surface of revolution $\mathcal{M}$ generated by C. We suppose that the diffusivity function does not depend on the angular variable θ, so that, abusing notation, we set $a(x(s, \theta)) = a(s)$.

Then regarding nonexistence of patterns to (1.1) a sufficient condition is found to be

$$K + (K_g)^2 \geqslant \frac{(a'\psi)'}{2a\psi} \quad \text{in } (0, l)$$

where K stands for the Gaussian curvature and K_g for the geodesic curvature of $\mathcal{M}$.

Note that this generalizes [4] since $\mathcal{M}$ with border under zero Neumann boundary condition is also allowed. This can be seen by taking $\psi \equiv 1$, which would correspond to a finite right circular cylinder, and then the nonexistence condition for patterns would read $a'' \leqslant 0$, as found in [4].

As for existence of patterns, after introducing a positive small parameter in the equation, we found that a sufficient condition is that the function

- $\sqrt{a}\psi$ has a isolated local minimum somewhere in $(0, l)$,

provided f is of bistable type and satisfies the equal-area condition ($f(u) = u - u^3$, for instance). In particular, if $a \equiv$ constant then the sufficient condition is satisfied as long as, roughly speaking, $\mathcal{M}$ has a neck.

The geometric profile of these patterns are also given. All these results remain true for a surface of revolution with border under Neumann boundary condition.

Many examples of surfaces satisfying both conditions, namely, for existence as well as for nonexistence of patterns are given.

This paper is divided as follows. In Section 2 we recall some material from stability of solution, differential geometry and function of bounded variation. In Section 3 we will extend the nonexistence result of patterns given in [1] to the case where a is nonconstant (see Remark 3.3(i)). In Section 4 we introduce a parameter $\epsilon > 0$ in the problem (1.1) and give sufficient conditions for existence of a family of stable stationary solution $\{v_\epsilon\}_{0<\epsilon<\epsilon_0}$, for some $\epsilon_0 > 0$, using Γ-convergence techniques.

In order to utilize Γ-convergence results f has to be a function of bistable type that satisfies the equal-area condition, sometimes also referred to as f being balanced. In Section 6 we prove that this condition is actually necessary in our approach.

2. Preliminaries

We begin with some definitions and known results from Differential Geometry which will be used in the following sections.

2.1. Surface of revolution

Consider $M = (\mathcal{M}, g)$ an n-dimensional Riemannian manifold with a metric given in local coordinates $x = (x^1, x^2, \ldots, x^n)$ given by (using Einstein summation convention)

$$ds^2 = g_{ij}\,dx^i\,dx^j, \qquad (g^{ij}) = (g_{ij}^{-1}), \qquad |g| = \det(g_{ij}).$$

Given a smooth vector field X on $\mathcal{M}$, the divergence operator of X is defined as

$$\operatorname{div}_g X = \frac{1}{\sqrt{|g|}} \frac{\partial}{\partial x^i}\left(\sqrt{|g|}X^i\right)$$

and the Riemannian gradient, denoted by ∇_g, of a sufficiently smooth real function ϕ defined on $\mathcal{M}$, as the vector field

$$(\nabla_g \phi)^i = g^{ij}\partial_j \phi.$$

We will see how the operator $\operatorname{div}_g(a(x)\nabla_g u)$ can be expressed for the particular case where $\mathcal{M}$ is a surface of revolution. Let C be the curve of $\mathbb{R}^3$ parametrized by