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Global attractivity in a population model with nonlinear death rate and distributed delays

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ABSTRACT

We study the global attractivity of the unique positive equilibrium of a population model with distributed delays and nonlinear death rate. Both delay dependent and delay independent criteria are obtained which generalize, unify and improve known criteria. These results will be applied to some models with bounded and unbounded death functions.

1. Introduction

In this paper, we investigate the global attractivity of first order differential equations of the form

$$x'(t)=p(t)\left[f\left(\int_{\tau(t)}^{t}h\big(x(s)\big)\,d_s\mu(t,s)\right)-g\big(x(t)\big)\right],\quad t\geqslant 0 \tag{1.1}$$

which model the growth of a single species population with distributed delays in the production (birth) rate f and nonlinear death (mortality) rate g. The presence of the function $p(t)$ makes it possible to model the growth when the environment parameters are changing proportionally with time. We assume that $p\in C[[0,\infty),(0,\infty)]$, $\tau\in C[[0,\infty),R]$ such that $\lim_{t\to\infty}\tau(t)=\infty$, $\tau(t)<t$ and

(C1) $\mu(t,s)$ is nondecreasing in s, continuous with respect to t and is normalized so that $\int_{\tau(t)}^{t}d_s\mu(t,s)=1$.
(C2) $f\circ h$, $h\in C[(a,\infty),(0,\infty)]$ and g is a positive continuous and increasing function on (a,∞) for some $a\geqslant-\infty$.

With each solution of (1.1) we associate an initial continuous function $\phi:[-\tau,0]\to(a,\infty)$ where $\tau=-\inf\{\tau(t)\colon t\in[0,\infty)\}>0$. The existence and uniqueness of a solution x associated with certain initial function ϕ can be proved using well known standard techniques; see for example the one used by [6] which in general requires the functions f,h to be Lipschitzian. Also, for some particular cases of (1.1) as the equation

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$$x'(t) = p(t)\left[f\left(\int_{\tau(t)}^{\beta(t)} h(x(s))\,d_s\bar{\mu}(t,s)\right) - \alpha x(t)\right], \tag{1.2}$$

where $\tau(t) \leqslant \beta(t) < t$ and all $t \in (0,\infty)$, a method of steps as in [19, Theorem 1.1.3] can be easily used to prove that its initial value problem has a unique solution without further assumptions on f, h other than their continuity. Therefore, we focus in this work only on the global attractivity of (1.1) bearing in mind that we deal only with continuously differentiable solutions on $[a,\infty)$.

Eq. (1.1) contains many important known general models from mathematical biology such as the non-autonomous balance equation

$$x'(t) = p(t)\big[H\big(x(t-\tau)\big) - g\big(x(t)\big)\big]. \tag{1.3}$$

The autonomous version of this equation has been derived by Blythe et al. [5] which in turn includes many important prototypes such as:

$$x'(t) = \frac{\beta_0 \theta^n}{\theta^n + x^n(t-\tau)} - \gamma x(t), \tag{1.4}$$

$$x'(t) = \frac{\beta_0 \theta^n x(t-\tau)}{\theta^n + x^n(t-\tau)} - \gamma x(t), \tag{1.5}$$

$$x'(t) = px(t-\tau)e^{-ax(t-\tau)} - \delta x(t), \tag{1.6}$$

and

$$x'(t) = pe^{-\gamma x(t-\tau)} - \mu x(t). \tag{1.7}$$

Eqs. (1.4) and (1.5) were used by Mackey and Glass [29] in order to describe some physiological control systems. Eq. (1.6) is the celebrated Nicholson blowflies model which was proposed by Gurney et al. [18]; while Eq. (1.7) was used by Wazewska-Czyzewska and Lasota [33] as a model of an erythropoietin system. For further results on the dynamical characteristics of the models (1.4)–(1.7) the reader is referred to [15,17,19,20,24,25,27,28].

Very little research was done on the global attractivity for equations with nonlinear death function in comparison with equations with linear one. For example; Freedman and Gopalsamy [14] studied the autonomous version of (1.3) which appears also as a special case of the model studied by Ivanov et al. [22]. Kuang [25] gave many global attractivity criteria for the equation

$$x'(t) = f\left(\int_{-\tau}^{-\sigma} x(t+s)\,d\mu(s)\right) - g\big(x(t)\big). \tag{1.8}$$

Recently, Qian [32] improved some of Kuang's results for the non-autonomous model (1.3). Arino et al. [1] and Berezansky et al. [2] studied two prototypes of (1.3) when g is quadratic.

In this work, we consider increasing death functions that can be bounded at infinity. Therefore the obtained results unify, generalize and improve some known global attractivity criteria; particularly our results improve the delay dependent criteria obtained by [6,22,25,32] and contain those of [1,2]. The obtained results will be applied on models with nonlinear death function such as a model mentioned by [3] which has the form

$$x'(t) = qx(t-\tau)e^{-x(t-\tau)} - g(x), \tag{1.9}$$

where g might have the form: $g(x) = \frac{\alpha x}{x+b}$ with $\alpha, b > 0$.

We will also be able to obtain sufficient criteria for the global attractivity of the positive equilibrium; say $\bar{y}$, of models of the form

$$y'(t) = p(t)y(t)\left[f\left(\int_{\tau(t)}^{t} h^*\big(y(s)\big)\,d_s\mu(t,s)\right) - g^*\big(y(t)\big)\right]. \tag{1.10}$$

In fact, using the transformation $y(t) = \bar{y}e^{x(t)}$ which transforms (1.10) to equation of the form (1.1) with $h(x(t)) = h^*(\bar{y}e^{x(t)})$ and $g(x(t)) = g^*(\bar{y}e^{x(t)})$, all our results can be restated to (1.10). This will enable us to obtain new global attractivity criteria for the non-autonomous Mackey–Glass model (see [29])

$$z'(t) = p(t)\left[a - b\frac{z(t)z^n(t-\tau)}{1+z^n(t-\tau)}\right], \quad t \geqslant 0,$$

by transforming it to an equation of the form (1.10).