J. Math. Anal. Appl. 410 (2014) 687–698

Contents lists available at ScienceDirect

Journal of Mathematical Analysis and Applications

www.elsevier.com/locate/jmaa

Layered viscosity solutions of nonautonomous Hamilton–Jacobi equations: Semiconvexity and relations to characteristics

Nguyen Hoang [a,*,1], Nguyen Mau Nam [b,2]

[a] Department of Mathematics, College of Education, Hue University, 32 LeLoi, Hue, Viet Nam
[b] Fariborz Maseeh Department of Mathematics and Statistics, Portland State University, PO Box 751, Portland, OR 97207, USA

ARTICLE INFO

Article history:
Received 6 May 2012
Available online 2 September 2013
Submitted by T. Witelski

Keywords:
Hamilton–Jacobi equation
Hopf-type formula
Layered viscosity solution
Semiconvexity

ABSTRACT

We construct an explicit representation of viscosity solutions of the Cauchy problem for the Hamilton–Jacobi equation (H,σ) on a given domain $\Omega=(0,T)\times\mathbb{R}^n$. It is known that, if the Hamiltonian $H=H(t,p)$ is not a convex (or concave) function in p, or $H(\cdot,p)$ may change its sign on $(0,T)$, then the Hopf-type formula does not define a viscosity solution on Ω. Under some assumptions for $H(t,p)$ on the subdomains $(t_i,t_{i+1})\times\mathbb{R}^n\subset\Omega$, we are able to arrange "partial solutions" given by the Hopf-type formula to get a viscosity solution on Ω. Then we study the semiconvexity of the solution as well as its relations to characteristics.

1. Introduction

This paper is devoted to constructing a representation formula for viscosity solutions of the Cauchy problem for the Hamilton–Jacobi equation (H,σ) of the form

$$\frac{\partial u}{\partial t}+H(t,x,D_xu)=0,\quad (t,x)\in\Omega=(0,T)\times\mathbb{R}^n, \tag{1.1}$$

$$u(0,x)=\sigma(x),\quad x\in\mathbb{R}^n. \tag{1.2}$$

The classical Hopf–Lax–Oleinik formula plays an important role in studying properties of solutions of problem (1.1)–(1.2), where $H=H(p)$. For general Hamiltonians $H=H(t,x,p)$, generalized solutions of the above problem have been established as value functions of the associated variational problems; see [1,5,9] and references therein. In most cases, Hamiltonians used to construct the formulas are concerned with the convexity (or concavity) in the global setting. This may meet requirements of calculus of variations or optimal control problems but in differential games, Hamiltonians are neither convex nor concave in general. In [3] the authors presented explicit estimates below and above of the form "maxmin" and "minmax" for the viscosity solutions where either the Hamiltonians or the initial data are the sum of a convex and a concave function. If these estimates are equal, then a representation formula for the solution is obtained; see also [4]. To best our knowledge, until now, no explicit representation formula of global solutions of Hamilton–Jacobi equations without concerning the convexity/concavity or its related versions has been constructed.

* Corresponding author.
E-mail addresses: nguyenhoanghue@gmail.com, nguyenhoang@hueuni.edu.vn (H. Nguyen), mau.nam.nguyen@pdx.edu (M.N. Nguyen).

[1] The research of Nguyen Hoang was partially supported by the NAFOSTED, Vietnam, under grant #101.01-2011.26 and by the project DHH2013-03-35.
[2] The research of Nguyen Mau Nam was partially supported by the Simons Foundation under grant #208785.

0022-247X/$ – see front matter
http://dx.doi.org/10.1016/j.jmaa.2013.08.047

We use the following notations which are standard in the field of nonlinear PDEs. For a positive real number T, denote $\Omega = (0, T) \times \mathbb{R}^n$. Let $|.|$ and $\langle .,. \rangle$ be the Euclidean norm and the scalar product in $\mathbb{R}^n$, respectively. For a function $u : \Omega \to \mathbb{R}$, we denote by $D_x u$ the gradient of u with respect to variable x, i.e., $D_x u = (u_{x_1}, \ldots, u_{x_n})$.

A continuous functions $u : [0, T] \times \mathbb{R}^n \to \mathbb{R}$ is called a *viscosity solution* of the Cauchy problem (1.1)–(1.2) on Ω provided that the following hold:

(i) $u(0, x) = \sigma(x)$ for all $x \in \mathbb{R}^n$;
(ii) For each $v \in C^1(\Omega)$, if $u - v$ has a local maximum at a point $(t_0, x_0) \in \Omega$, then

$$v_t(t_0, x_0) + H\big(t_0, x_0, D_x v(t_0, x_0)\big) \leqslant 0, \tag{1.3}$$

and if $u - v$ has a local minimum at a point $(t_0, x_0) \in \Omega$, then

$$v_t(t_0, x_0) + H\big(t_0, x_0, D_x v(t_0, x_0)\big) \geqslant 0. \tag{1.4}$$

If the continuous function u satisfies (i) and (ii)-(1.3) (resp. (ii)-(1.4)), then it is called a viscosity subsolution (resp. supersolution) of the problem (1.1)–(1.2). A such C^1-function v in the definition is called a test function for the viscosity solution.

Note that the notion of viscosity solution of Hamilton–Jacobi equation was originally presented by M.G. Crandall and P.L. Lions [7]. The prominent monograph of P.L. Lions [10] has a great impact in the development of theory of viscosity solutions. In [2] M. Bardi and L.C. Evans proved that Hopf–Lax and Hopf formulas are viscosity solutions of the problem (1.1)–(1.2) where $H = H(p)$ or σ is convex. These are the first results on the explicit representation of the viscosity solutions of Hamilton–Jacobi equations.

In this paper, we consider nonautonomous Hamilton–Jacobi equations in a new situation. We suppose that the behavior of the Hamiltonian $H(t, x, p)$ may vary on each subdomain $[t_i, t_{i+1}] \times \mathbb{R}^n \times \mathbb{R}^n$ of the domain $[0, T] \times \mathbb{R}^n \times \mathbb{R}^n$, where

$$0 = t_1 < t_2 < \cdots < t_k = T.$$

For example, $H(t, x, \cdot)$ may be convex for all $(t, x) \in [t_{i-1}, t_i] \times \mathbb{R}^n$, and then $H(t, x, \cdot)$ changes into a concave function in the successive subdomain $(t, x) \in [t_i, t_{i+1}] \times \mathbb{R}^n$.

The paper is structured as follows. In Section 2, we introduce a way to joint viscosity solutions of problem (1.1)–(1.2) on subregions of the form $[t_i, t_{i+1}] \times \mathbb{R}^n$ in order to obtain a global solution called a *"layered viscosity solution"* of the problem as a whole. The proof is based on a key technique called "extrema at the terminal time", an important lemma in [8,6]. In Section 3, we examine the case where the Hamiltonian $H = H(t, p)$ is not necessarily convex or concave but the initial data $\sigma(x)$ is convex. Under some assumptions on the given data, we use the Hopf-type formula for "partial solutions" on subdomain $[t_i, t_{i+1}] \times \mathbb{R}^n$

$$u_i(t, x) = \max_{q \in \mathbb{R}^n} \left\{ \langle x, q \rangle - \sigma_i^*(q) - \int_{t_i}^{t} H(\tau, q)\, d\tau \right\},$$

where σ_i^* is the Fenchel conjugate of the convex initial data σ_i, to build a layered viscosity solution of the Cauchy problem on the whole domain. We also prove that the obtained viscosity solution inherits the semiconvexity analogous to "partial solutions". In Section 4, we study relations between the Hopf-type formula and the layered viscosity solution as well as its singularity based on the semiconvexity of the solution $u(t, x)$ in connection with the characteristics.

2. Layered viscosity solution in general cases

Consider the nonlinear first order partial differential equation $F(x, w, Dw) = 0$ on a domain $\mathcal{O} \subset \mathbb{R}^m$, where $F : \mathcal{O} \times \mathbb{R} \times \mathbb{R}^m \to \mathbb{R}$ is a continuous function. Suppose that $\mathcal{O}$ can be divided in two open subsets $\mathcal{O}_1$ and $\mathcal{O}_2$ by a C^1 surface Γ, that is $\mathcal{O} = \mathcal{O}_1 \cup \mathcal{O}_2 \cup \Gamma$, then several compatible conditions can be added on $\mathcal{O}_i$ and Γ, as well as on C^1-solutions w_i of the equation $F(x, w, Dw) = 0$ on $\mathcal{O}_i \cup \Gamma$ for $i = 1, 2$, to get a viscosity solution on entire $\mathcal{O}$; see [6, Theorem 1.3]. However, in a specific case of the Cauchy problem, the situation seems to be rather simple as follows.

Theorem 2.1. *Let $H(t, x, p)$ be a continuous function on $[0, T] \times \mathbb{R}^n \times \mathbb{R}^n$ and let $\sigma_1(x), \sigma_2(x)$ be continuous functions on $\mathbb{R}^n$. For $t_1 \in (0, T)$, consider the following Cauchy problems*:

$$\begin{cases} u_t + H(t, x, D_x u) = 0, & (t, x) \in \Omega_1 = (0, t_1) \times \mathbb{R}^n, \\ u(0, x) = \sigma_1(x), & x \in \mathbb{R}^n, \end{cases} \tag{2.1}$$

and

$$\begin{cases} u_t + H(t, x, D_x u) = 0, & (t, x) \in \Omega_2 = (t_1, T) \times \mathbb{R}^n, \\ u(t_1, x) = \sigma_2(x), & x \in \mathbb{R}^n. \end{cases} \tag{2.2}$$