J. Math. Anal. Appl. 410 (2014) 764–770

Contents lists available at ScienceDirect

Journal of Mathematical Analysis and Applications

www.elsevier.com/locate/jmaa

Algebraic independence of logarithmic singularities of some complex functions

Maciej Radziejewski

Faculty of Mathematics and Computer Science, Adam Mickiewicz University, ul. Umultowska 87, PL-61-614 Poznań, Poland

ARTICLE INFO

Article history:
Received 23 August 2013
Available online 9 September 2013
Submitted by Richard M. Aron

Keywords:
Algebraic independence
Singularities
Complex functions
Dirichlet series
Selberg class

ABSTRACT

We show that algebraic independence of some complex functions of one variable over regular functions implies their algebraic independence over a larger ring, containing complex powers of regular functions. Based on this we obtain a generalization of a special case of the theorem of Kaczorowski and Perelli on functional independence of logarithms of functions in the Selberg class. As an application we state a new result on oscillations of arithmetical functions.

© 2013 Elsevier Inc. All rights reserved.

1. Introduction

We show an analytical result about algebraic independence of complex functions, motivated by applications in quantitative factorizations theory, specifically in the study of irregularities in the distribution of algebraic integers with unique length of factorization into irreducibles. The extension of the notion of a number through history has placed algebraic integers among the central objects of interest to number theory. Factorization into irreducible integers in an algebraic number field K (a finite extension of the rationals) need not be unique, and in fact even the length of such a factorization need not be unique: Carlitz [1] has shown that it is unique for all integers of K if and only if the class number of K equals 1 or 2. Quantitative properties of various subsets of algebraic integers with prescribed factorization properties (e.g., irreducibles, numbers with unique factorization, numbers with unique factorization length) were studied by many authors starting with Fogels [2] and then Narkiewicz who answered a question of Turán [9]. Given a subset A of all integers in K one studies the asymptotic behaviour of its counting function $A(x)$ denoting the number of non-associated elements $a \in A$ with norm $N(a)$ not exceeding $x > 0$. The main term of $A(x)$ can often be defined analytically, as in Kaczorowski [3] and Kaczorowski and Perelli [6]. This is possible for any A such that the zeta function of A, i.e. the function

$$Z(s, A) = \sum_{a \in A}{}' N(a)^{-s}, \quad \Re s > 1,$$

where the summation is over non-associated elements, has an appropriate extension. Such a zeta function is generally regular in the half-plane $\Re s > 1$. Its extension to, say, the half-plane $\Re s > \frac{1}{3}$, depends on the properties defining A and usually involves representing $Z(s, A)$ as an expression in the logarithms and complex powers of well-behaved functions, i.e. L-functions belonging to the Selberg class $\mathcal{S}$ of L-functions, cf. [15,7,4]. For the purposes of this paper it is not necessary to recall the definition of the Selberg class. We only note that all elements $G(s)$ of $\mathcal{S}$ are complex functions meromorphic on $\mathbf{C}$, regular for $s \neq 1$, non-zero in the half-plane $\Re s > 1$, and they satisfy $\lim_{s \to +\infty} G(s) = 1$. Typically the aforementioned

E-mail address: maciejr@amu.edu.pl.

0022-247X/$ – see front matter © 2013 Elsevier Inc. All rights reserved.
http://dx.doi.org/10.1016/j.jmaa.2013.09.005

expression of $Z(s, A)$ involves complicated function coefficients that are known, e.g., to be regular in a larger half-plane. Therefore $Z(s, A)$ is expected to have singularities at the zeros of the L-functions involved, but it is not obvious that it does have any. Roughly speaking, the main term of $A(x)$ depends on the singularities of $Z(s, A)$ on the real line [3] while the existence of a singularity ρ off the real line contributes to the understanding of the error term, defined as the difference of $A(x)$ and the main term. One can usually show that the error term $E(x)$ is subject to oscillations of logarithmic frequency and size $x^{\Re\rho}$ times some extra factor, meaning that there exists a sequence (x_n) increasing to infinity such that the signs of $E(x_n)$ alternate, $|E(x_n)|$ is roughly of the size $x^{\Re\rho}$ (times the extra factor), $\limsup_{n\to\infty} \log x_n/n < +\infty$ and $\lim_{n\to\infty} \log x_{n+1}/\log x_n = 1$, cf., e.g., [8,5,13].

Narkiewicz [9] obtained an upper bound for the counting function of the set G_k of algebraic integers (in a suitable field) with at most k different lengths of factorizations into irreducibles, providing a quantitative strenghtening of the result of Carlitz. Śliwa [16] studied the asymptotic main term of $G_k(x)$ and Kaczorowski [3] obtained more precise asymptotics with an upper bound for the error term. Cf. also [14] for a more complete account. Oscillations of the error term were investigated by the author [10,12] and by Schmid and the author [14] who were able to show the existence of oscillations in the case $k \geqslant 2$. For $k = 1$ a result similar to Theorem 3 below (with oscillations of size $x^{1/2-\varepsilon}$) was only shown in special cases or by assuming some unproved conjectures.

The applications in analytic number theory (through Dirichlet series) are the reason why we consider rather specific branching of functions. Results analogous to Theorem 1 below can also be proved in greater generality, however their formulation would be more complex.

2. Statement of results

We call a set $\mathcal{D} \subseteq \mathbf{C}$ a region if it is non-empty, open and connected (hence arc-wise connected). If G is a function meromorphic in the neighbourhood of ρ we let $m(\rho, G)$ denote the order of a zero of G at ρ, with $m(\rho, G) = -m$ in case of a pole of order m and $m(\rho, G) = 0$ if G is regular and non-zero at ρ. The letter s stands for a complex variable throughout. As usual, we denote by $\mathbf{N}$, $\mathbf{Z}$, $\mathbf{Q}$, $\mathbf{R}$ and $\mathbf{C}$ the sets of positive integer, integer, rational, real and complex numbers, respectively, and by $\mathbf{N}_0$ the set of non-negative integers.

Let $\mathcal{D} \subseteq \mathbf{C}$ be a region and let $C \subseteq \mathcal{D}$ be a discrete closed set contained in a certain half-plane:

$$\bigvee_{\sigma_0\in\mathbf{R}} \bigwedge_{s\in C} \Re s \leqslant \sigma_0. \tag{1}$$

We say that a complex function F is defined in $\mathcal{D}$ with possible singularities in C if

(i) the function F is defined and regular on $\mathcal{D}$ apart from horizontal cuts inside $\mathcal{D}$, extending from the points $\rho \in C$ to the left, up to the boundary of $\mathcal{D}$ or to $\rho - \infty$, and
(ii) F has an extension (possibly branched) to $\mathcal{D} \setminus C$ covering all points inside the cuts at least twice (from “above” and from “below”).

Let $\mathrm{Br}(\mathcal{D}; C)$ denote the set of all such functions. We say that $F \in \mathrm{Br}(\mathcal{D}; C)$ has a singularity at a point $\rho \in C$ if F has no extension in $\mathrm{Br}(\mathcal{D}; C \setminus \{\rho\})$. If F has a singularity at all points $\rho \in C$ we call F maximal. Let $\mathrm{Br}(\mathcal{D})$ denote the set of all maximal functions defined in $\mathcal{D}$ with possible singularities. We show in Section 3 that essentially we can treat all functions with possible singularities as maximal (Lemma 5), and that $\mathrm{Br}(\mathcal{D})$ is an integral domain (Lemma 6).

By a branch of $G(s)^w$ we mean any function of the form $e^{wh(s)}$, where $h(s)$ is a function defined in $\mathcal{D}$ with possible singularities such that we have $e^{h(s)} = G(s)$ identically. In expressions of the form $\log(s-\rho)$ or $(s-\rho)^w$, $\rho, w \in \mathbf{C}$, we generally mean the principal branches. The cases when we take arbitrary branches are clearly indicated. The same applies in the case of $\log G(s)$ if G is a function from the Selberg class.

We denote by $\mathrm{Hol}(\mathcal{D})$ the ring of functions holomorphic in $\mathcal{D}$ and by $\mathrm{Hol}^{\mathbf{C}}(\mathcal{D})$ the subring of $\mathrm{Br}(\mathcal{D})$ generated by $\mathrm{Hol}(\mathcal{D})$ and by (all) branches of functions of the form $G(s)^w$ where G is a function regular in $\mathcal{D}$ whose set of zeros C satisfies (1), and $w \in \mathbf{C}$. Finally, let $\mathrm{LI}(\mathcal{D})$ (for “logarithms and integer powers”) be the subring of $\mathrm{Br}(\mathcal{D})$ consisting of the functions with all singularities of the form

$$F(s) = \sum_{j=1}^{m} (s-\rho)^{-b_j} \bigl(\log(s-\rho)\bigr)^{c_j} h_j(s),$$

where b_js and c_js are integers and each h_j is regular in the neighbourhood of ρ. We show the following.

Theorem 1. *Let $\mathcal{D} \subseteq \mathbf{C}$ be a region and let $F_1, \dots, F_n \in \mathrm{LI}(\mathcal{D})$ be algebraically independent over* $\mathrm{Hol}(\mathcal{D})$. *Then $F_1, \dots, F_n$ are algebraically independent over* $\mathrm{Hol}^{\mathbf{C}}(\mathcal{D})$.

Let $F_1, \dots, F_n \in \mathcal{S}$ be such that $\log F_1, \dots, \log F_n$ are linearly independent over $\mathbf{Q}$. Let $T > 0$ and let $\mathcal{D}$ be a region containing the set

Download English Version:

https://daneshyari.com/en/article/4616276

Download Persian Version:

https://daneshyari.com/article/4616276

Daneshyari.com