J. Math. Anal. Appl. 410 (2014) 807–826

Contents lists available at ScienceDirect

Journal of Mathematical Analysis and Applications

www.elsevier.com/locate/jmaa

Formal adjoint operators and asymptotic formula for solutions of autonomous linear integral equations

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A R T I C L E I N F O

Article history:
Received 29 March 2013
Available online 28 August 2013
Submitted by X. Zhang

Keywords:
Integral equations
Phase space
Generator of solution semigroup
Formal adjoint operator

A B S T R A C T

For autonomous linear integral equations, we establish an explicit asymptotic representation formula of solutions, developing the spectral analysis for the generator of the solution semigroup as well as the one for the formal adjoint operator associated with the generator.

1. Introduction

In this paper we are concerned with the linear integral equation with infinite delay

$$x(t) = \int_{-\infty}^{t} K(t-s)x(s)\,ds, \tag{E}$$

where the kernel K is a measurable $m \times m$ matrix valued function with complex components. In a special case of $K(t) = 0$ a.e. on an interval $[h, \infty)$, $h < \infty$, Eq. (E) is reduced to the equation

$$x(t) = \int_{t-h}^{t} K(t-s)x(s)\,ds$$

which is an equation with finite delay h. For various integral equations with infinite delay including equations with finite delay, numerous results are obtained by many investigators; for detailed informations on the aspects, we refer the reader to the book [3] and the references therein.

Diekmann and Gyllenberg [2] have recently treated integral equations with infinite delay in a dynamical-systems approach and established several remarkable results including the principle of linearized stability for integral equations. Motivated by the paper [2] which may be seen as a pioneering paper, several studies for integral equations in the approach

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0022-247X/$ – see front matter
http://dx.doi.org/10.1016/j.jmaa.2013.08.035

are now proceeded; e.g. [7,8]. In the present paper we subsequently treat Eq. (E), and analyze the asymptotic behavior of solutions of Eq. (E), based on the theory of semigroups of operators. In particular, assuming the condition

$$\int_0^\infty \|K(t)\| e^{\rho t}\, dt < \infty \quad \text{and} \quad \operatorname{ess\,sup}\{\|K(t)\| e^{\rho t} \mid t \geqslant 0\} < \infty,$$

for a constant $\rho > 0$ (which is imposed on (E) throughout the paper), we set up Eq. (E) as a linear (autonomous) functional equation on the phase space $X := L^1_\rho(\mathbb{R}^-; \mathbb{C}^m)$, $\mathbb{R}^- := (-\infty, 0]$, which is a complex Banach space of all the equivalent classes of functions

$$\phi : \mathbb{R}^- \to \mathbb{C}^m \quad \text{and} \quad \phi(\theta) e^{\rho\theta} \text{ is integrable on } \mathbb{R}^-,$$

equipped with norm

$$\big(\|\phi\| :=\big)\ \|\phi\|_{1,\rho} = \int_{-\infty}^0 |\phi(\theta)| e^{\rho\theta}\, d\theta, \quad \phi \in X,$$

and by developing spectral analysis for the generator of the solution semigroup associated with the linear functional equation, we establish an explicit asymptotic formula (Theorem 3.2) for solutions of Eq. (E). It should be noticed that relation (3.59) in [2] also gives a nice formula for the asymptotic behavior of solution of Eq. (E). We emphasize that our asymptotic formula (Theorem 3.2) is written in a more explicit form than relation (3.59) in [2]. Each term in our formula is computable systematically; indeed, applying Theorem 3.2 we establish more explicit asymptotic results for solutions of some concrete equations (Examples 4.1 and 4.2).

Our technique employed here is similar to the standard one in the theory of functional differential equations (e.g. [4, Chapter 7] and [5, Chapter 5]). For the obtainment of an analogous result available to the analysis of integral equations, we need further to develop the theory, which is often referred to as the formal adjoint theory, to establish an explicit form of the projection on a certain closed subspace of the phase space. For background on the formal adjoint theory of functional differential equations, see, e.g. [4, Section 7.5] and [11]; also, see [9,10,12] for the one in difference equations with delay. In Section 3.4, introducing an operator (named the formal adjoint operator) which is a dual operator of the generator of the solution semigroup with respect to a certain bilinear form, we provide a representation form of the projection which is explicitly written by utilizing a part of some bases belonging to the phase space, together with a part of some bases concerned with the formal adjoint operator (Theorem 3.1). As in the case of ordinary differential equations (e.g. [1]), it may be expected that the representation form of the projection plays an essential role when one attempts to solve stability problems in nonlinear integral equations by establishing center manifold theory.

2. Notations and some preparatory results

Let $\mathbb{N}$, $\mathbb{R}^+$, $\mathbb{R}^-$, $\mathbb{R}$ and $\mathbb{C}$ be the set of natural numbers, nonnegative real numbers, nonpositive real numbers, real numbers and complex numbers, respectively. For an $m \in \mathbb{N}$, we denote by $\mathbb{C}^m$ the space of all m-column vectors whose components are complex numbers, with the Euclidean norm $|\cdot|$.

Given Banach space $(U, \|\cdot\|)$, we denote by $\mathcal{L}(U)$ the space of bounded linear operators on U with the norm

$$\|Q\| := \sup\{\|Q(u)\| \mid u \in U,\ \|u\| = 1\}$$

for $Q \in \mathcal{L}(U)$. In particular, for an $m \times m$ matrix M with complex components, $\|M\|$ means its operator norm regarding M as an operator on $\mathbb{C}^m$.

In this section, following the paper [7] mainly, we will present several preparatory results on linear integral equations which are needed for our later arguments.

2.1. Phase space and initial value problems

Let ρ be a fixed positive constant, and denote by X the function space

$$L^1_\rho(\mathbb{R}^-; \mathbb{C}^m) := \{\phi : \mathbb{R}^- \to \mathbb{C}^m \mid \phi(\theta) e^{\rho\theta} \text{ is integrable on } \mathbb{R}^-\}.$$

Clearly, X is a Banach space endowed with norm

$$\|\phi\| := \int_{-\infty}^0 |\phi(\theta)| e^{\rho\theta}\, d\theta, \quad \phi \in X.$$