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Two-term trace estimates for relativistic stable processes

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ABSTRACT

We prove trace estimates for the relativistic α-stable process extending the result of Bañuelos and Kulczycki (2008) in the stable case.

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1. Introduction and statement of main results

For $m>0$, an $\mathbb{R}^d$-valued process with independent, stationary increments having the following characteristic function

$$\mathbb{E}e^{i\xi\cdot X_t^{\alpha,m}} = e^{-t\{(m^{2/\alpha}+|\xi|^2)^{\alpha/2}-m\}}, \quad \xi\in\mathbb{R}^d,$$

is called relativistic α-stable process with mass m. We assume that sample paths of $X_t^{\alpha,m}$ are right continuous and have left-hand limits a.s. If we put $m=0$ we obtain the symmetric rotation invariant α-stable process with the characteristic function $e^{-t|\xi|^\alpha}$, $\xi\in\mathbb{R}^d$. We refer to this process as isotropic α-stable Lévy process. For the rest of the paper we keep α, m and $d\geqslant 2$ fixed and drop α, m in the notation, when it does not lead to confusion. Hence from now on the relativistic α-stable process is denoted by X_t and its counterpart isotropic α-stable Lévy process by $\widetilde{X}_t$. We keep this notational convention consistently throughout the paper, e.g., if $p_t(x-y)$ is the transition density of X_t, then $\tilde{p}_t(x-y)$ is the transition density of $\widetilde{X}_t$.

In Ryznar [11] Green function estimates of the Schödinger operator with the free Hamiltonian of the form

$$\left(-\Delta+m^{2/\alpha}\right)^{\alpha/2}-m,$$

were investigated, where $m>0$ and Δ is the Laplace operator acting on $L^2(\mathbb{R}^d)$. Using the estimates in Lemma 2.6 below and proof in Bañuelos and Kulczycki (2008) we provide an extension of the asymptotics in [3] to the relativistic α-stable processes for any $0<\alpha<2$.

Brownian motion has characteristic function

$$\mathbb{E}^0 e^{i\xi\cdot B_t} = e^{-t|\xi|^2}, \quad \xi\in\mathbb{R}^d.$$

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Let $\beta = \alpha/2$. Ryznar showed that X_t can be represented as a time-changed Brownian motion. Let $T_\beta(t)$, $t > 0$, denote the strictly β-stable subordinator with the following Laplace transform

$$\mathbb{E}^0 e^{-\lambda T_\beta(t)} = e^{-t\lambda^\beta}, \quad \lambda > 0. \tag{1.1}$$

Let $\theta_\beta(t,u)$, $u > 0$, denote the density function of $T_\beta(t)$. Then the process $B_{T_\beta(t)}$ is the standard symmetric α-stable process.

Ryznar [11, Lemma 1] showed that we can obtain $X_t = B_{T_\beta(t,m)}$, where a subordinator $T_\beta(t,m)$ is a positive infinitely divisible process with stationary increments with probability density function

$$\theta_\beta(t,u,m) = e^{-m^{1/\beta}u+mt}\theta_\beta(t,u), \quad u > 0.$$

Transition density of $T_\beta(t,m)$ is given by $\theta_\beta(t,u-v,m)$. Hence the transition density of X_t is $p(t,x,y) = p(t,x-y)$ given by

$$p(t,x) = e^{mt}\int_0^\infty \frac{1}{(4\pi u)^{d/2}} e^{\frac{-|x|^2}{4u}} e^{-m^{1/\beta}u}\theta_\beta(t,u)\,du. \tag{1.2}$$

Then

$$p(t,x,x) = p(t,0) = e^{mt}\int_0^\infty \frac{1}{(4\pi u)^{d/2}} e^{-m^{1/\beta}u}\theta_\beta(t,u)\,du.$$

The function $p(t,x)$ is a radially symmetric decreasing and that

$$p(t,x) \leqslant p(t,0) \leqslant e^{mt}\int_0^\infty \frac{1}{(4\pi u)^{d/2}}\theta_\beta(t,u)\,du = e^{mt}t^{-d/\alpha}\frac{\omega_d\Gamma(d/\alpha)}{(2\pi)^d\alpha}, \tag{1.3}$$

where $\omega_d = \frac{2\pi^{d/2}}{\Gamma(d/2)}$ is the surface area of the unit sphere in $\mathbb{R}^d$. For an open set D in $\mathbb{R}^d$ we define the first exit time from D by $\tau_D = \inf\{t \geqslant 0\colon\ X_t \notin D\}$.

We set

$$r_D(t,x,y) = \mathbb{E}^x\big[p(t-\tau_D, X_{\tau_D}, y);\ \tau_D < t\big], \tag{1.4}$$

and

$$p_D(t,x,y) = p(t,x,y) - r_D(t,x,y), \tag{1.5}$$

for any $x, y \in \mathbb{R}^d$, $t > 0$. For a nonnegative Borel function f and $t > 0$, let

$$P_t^D f(x) = \mathbb{E}^x\big[f(X_t)\colon\ t < \tau_D\big] = \int_D p_D(t,x,y) f(y)\,dy,$$

be the semigroup of the killed process acting on $L^2(D)$, see, Ryznar [11, Theorem 1].

Let D be a bounded domain (or of finite volume). Then the operator P_t^D maps $L^2(D)$ into $L^\infty(D)$ for every $t > 0$. This follows from (1.3), (1.4), and the general theory of heat semigroups as described in [7]. It follows that there exists an orthonormal basis of eigenfunctions $\{\varphi_n\colon\ n = 1,2,3,\ldots\}$ for $L^2(D)$ and corresponding eigenvalues $\{\lambda_n\colon\ n = 1,2,3,\ldots\}$ of the generator of the semigroup P_t^D satisfying

$$\lambda_1 < \lambda_2 \leqslant \lambda_3 \leqslant \cdots,$$

with $\lambda_n \to \infty$ as $n \to \infty$. By definition, the pair $\{\varphi_n, \lambda_n\}$ satisfies

$$P_t^D \varphi_n(x) = e^{-\lambda_n t}\varphi_n(x), \quad x \in D,\ t > 0.$$

Under such assumptions we have

$$p_D(t,x,y) = \sum_{n=1}^\infty e^{-\lambda_n t}\varphi_n(x)\varphi_n(y). \tag{1.6}$$

In this paper we are interested in the behavior of the trace of this semigroup

$$Z_D(t) = \int_D p_D(t,x,x)\,dx. \tag{1.7}$$