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Irreducible representations of free products of matrix algebras

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ABSTRACT

In this paper, we give a characterization of irreducible representation of $*_{i=1}^{m} M_{n_i}(\mathbb{C})$ into $M_{kN}(\mathbb{C})$, where $k \in \mathbb{N}$ and the least common multiple N of $n_1, \ldots, n_m$ is finite. We also give a necessary condition for the existence of irreducible representation of $*_{i=1}^{m} M_{n_i}(\mathbb{C})$ into $M_{kN}(\mathbb{C})$.

1. Introduction

A C^*-algebra $\mathcal{U}$ is a complex Banach algebra with an involution that satisfies the additional condition $\|T^*T\| = \|T\|^2$ for $T \in \mathcal{U}$. Every C^*-algebra can be viewed as a normed-closed self-adjoint subalgebra of $B(H)$.

By a representation of a C^*-algebra $\mathcal{U}$ on a Hilbert space H, we mean a $*$-homomorphism ρ from $\mathcal{U}$ into $B(H)$. In addition, ρ is one-to-one (hence, a $*$-isomorphism), it is described as a faithful representation. The Gelfand–Naimark theorem shows that every C^*-algebra has a faithful representation on some Hilbert space.

Suppose that $\mathcal{U}$ is a C^*-algebra and that φ and ψ are representations of $\mathcal{U}$ on Hilbert spaces H and K, respectively. We say that φ and ψ are (unitarily) equivalent if there is an isomorphism U from H onto K such that $\psi(A) = U\varphi(A)U^*$ for any A in $\mathcal{U}$. (For more on the operator theory, see [10,11,14].)

The reduced free product of C^*-algebras with specified states on them was introduced independently by Avitzour [1] and Voiculescu [15].

For the basic properties of free probability theory, we refer the reader to [5,13,17].

Using the free probability theory, Voiculescu [16] proved that the free group von Neumann algebra does not have Cartan subalgebras and solved a longstanding question in finite von Neumann algebra. Many other powerful applications of free probability to questions on operator algebras are due to Ge, Dykema, Shlyakhenko and other mathematicians. (See [2,4,7,8].)

There are many literatures studying the structure of reduced free products of C^*-algebras. In [3,6], Dykema studied the reduced free product of a finite family of finite dimensional abelian C^*-algebras and the von Neumann algebra free products of finite dimensional von Neumann algebras. In [9], Ivanov found a necessary and sufficient condition for the simplicity and uniqueness of trace for reduced free products of finite families of finite dimensional C^*-algebras with specified traces on them.

Recently, several authors considered the representations of the free product of C^*-algebras. In [12], the authors gave a characterization of the irreducible representations of the free product $M_2(\mathbb{C}) * M_2(\mathbb{C})$.

In this paper, we will consider the representations of the free product of matrix algebras.

In Section 2, we characterize the representation of $M_n(\mathbb{C})$ into a C^*-algebra $\mathcal{U}$. We also give an explicit expression of a representation of $M_n(\mathbb{C})$ into $M_{kn}(\mathbb{C})$.

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In Section 3, we consider the representation of $*_{i=1}^{m} M_n(\mathbb{C})$ into $M_{kn}(\mathbb{C})$. We give a characterization of irreducible representation of $*_{i=1}^{m} M_n(\mathbb{C})$ into $M_{kn}(\mathbb{C})$ and give a necessary condition for the existence of irreducible representation of $*_{i=1}^{m} M_n(\mathbb{C})$ into $M_{kn}(\mathbb{C})$.

In Section 4, we consider the case of representations of $*_{i=1}^{m} M_{n_i}(\mathbb{C})$ into $M_{kN}(\mathbb{C})$, where N is the least common multiple of $n_1, \ldots, n_m$. We give a characterization of irreducible representation of $*_{i=1}^{m} M_{n_i}(\mathbb{C})$ into $M_{kN}(\mathbb{C})$ and give a necessary condition for the existence of irreducible representation of $*_{i=1}^{m} M_{n_i}(\mathbb{C})$ into $M_{kN}(\mathbb{C})$.

2. Representations of $M_n(\mathbb{C})$ $(n \geqslant 2)$

Theorem 2.1. *Let $\mathcal{U}$ be a unital C^*-algebra and $\{T_i\}_{i=1}^{n-1} \subseteq \mathcal{U}$. Then the following conditions are equivalent.*

(1) *There is a unital $*$-homomorphism $\pi : M_n(\mathbb{C}) \to \mathcal{U}$ such that $\pi(e_{i,i+1}) = T_i$, $i = 1, 2, \ldots, n-1$;*
(2) $T_1T_1^* + \sum_{i=1}^{n-1} T_i^*T_i = I$, $T_i^*T_i = T_{i+1}T_{i+1}^*$ *for* $i = 1, \ldots, n-2$, $T_iT_j = 0$ *for* $j \neq i+1$ *and* $T_iT_j^* = 0$ *for* $j < i$.

Proof. (1) ⇒ (2): If there is a unital $*$-homomorphism $\pi : M_n(\mathbb{C}) \to \mathcal{U}$ such that $\pi(e_{i,i+1}) = T_i$, $i = 1, 2, \ldots, n-1$, then

$$T_iT_j = \pi(e_{i,i+1})\pi(e_{j,j+1}) = \pi(e_{i,i+1}e_{j,j+1}) = 0$$

for $j \neq i+1$,

$$T_iT_j^* = \pi(e_{i,i+1})\big(\pi(e_{j,j+1})\big)^* = \pi(e_{i,i+1})\pi(e_{j+1,j}) = \pi(e_{i,i+1}e_{j+1,j}) = 0$$

for $j < i$,

$$\begin{aligned} T_i^*T_i &= \pi\big(e_{i,i+1}^*\big)\pi(e_{i,i+1}) = \pi(e_{i+1,i}e_{i,i+1}) = \pi(e_{i+1,i+1}) = \pi(e_{i+1,i+2}e_{i+2,i+1}) \\ &= \pi(e_{i+1,i+2})\big(\pi(e_{i+1,i+2})\big)^* = T_{i+1}T_{i+1}^* \end{aligned}$$

for $i = 1, \ldots, n-2$ and

$$T_1T_1^* + \sum_{i=1}^{n-1} T_i^*T_i = \pi(e_{1,1}) + \sum_{i=1}^{n-1} \pi(e_{i+1,i+1}) = \pi\left(e_{1,1} + \sum_{i=1}^{n-1} e_{i+1,i+1}\right) = \pi(I) = I_{\mathcal{U}}.$$

(2) ⇒ (1): Define $\pi : M_n(\mathbb{C}) \to \mathcal{U}$ by

$$\pi\big((a_{i,j})_{n\times n}\big) = \sum_{j>i} a_{i,j}T_i \cdots T_{j-1} + \sum_{j<i} a_{i,j}T_{i-1}^* \cdots T_j^* + a_{1,1}T_1T_1^* + \sum_{i=2}^{n-1} a_{i,i}T_i^*T_i$$

for any $(a_{i,j})_{n\times n} \in M_n(\mathbb{C})$.

Thus

$$\begin{aligned} &\pi(e_{1,1}) = T_1T_1^*, \quad \pi(e_{1,2}) = T_1, \quad \ldots, \quad \pi(e_{1,n}) = T_1 \cdots T_{n-1}, \\ &\pi(e_{2,1}) = T_1^*, \quad \pi(e_{2,2}) = T_1^*T_1 = T_2T_2^*, \quad \ldots, \quad \pi(e_{2,n}) = T_2 \cdots T_{n-1}, \\ &\pi(e_{3,1}) = T_2^*T_1^*, \quad \pi(e_{3,2}) = T_2^*, \quad \ldots, \quad \pi(e_{3,n}) = T_3 \cdots T_{n-1}, \\ &\quad \vdots \\ &\pi(e_{n,1}) = T_{n-1}^* \cdots T_1^*, \quad \pi(e_{n,2}) = T_{n-1}^* \ldots T_2^*, \quad \ldots, \quad \pi(e_{n,n}) = T_{n-1}^*T_{n-1}. \end{aligned}$$

It is easy to check that π is a unital $*$-homomorphism and $\pi(e_{i,i+1}) = T_i$. □

Thus every unital $*$-homomorphism $\pi : M_n(\mathbb{C}) \to B(H)$ is determined by some $T_1, \ldots, T_{n-1}$ in $B(H)$.

Theorem 2.2. *Suppose that $\{T_i\}_{i=1}^{n-1} \subseteq B(H)$. Then the following conditions are equivalent.*

(1) $T_1T_1^* + \sum_{i=1}^{n-1} T_i^*T_i = I$, $T_i^*T_i = T_{i+1}T_{i+1}^*$ *for* $i = 1, \ldots, n-2$, $T_iT_j = 0$ *for* $j \neq i+1$ *and* $T_iT_j^* = 0$ *for* $j < i$.
(2) *There exists an orthonormal basis $\bigcup_{k=1}^{n}\{e_i^{(k)},\ i \in I\}$ of H such that $T_k(e_i^{(k+1)}) = e_i^{(k)}$, $k = 1, \ldots, n-1$ and $T_m(e_i^{(k)}) = 0$, $k \neq m+1$, $k = 1, \ldots, n$; $m = 1, \ldots, n-1$ and $i \in I$ where I is an index set.*