J. Math. Anal. Appl. 410 (2014) 908–917

Contents lists available at ScienceDirect

Journal of Mathematical Analysis and Applications

www.elsevier.com/locate/jmaa

Global solutions to a chemotaxis system with non-diffusive memory

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ARTICLE INFO

Article history:
Received 11 April 2013
Available online 9 September 2013
Submitted by K. Nishihara

Keywords:
Chemotaxis system
Global solutions

ABSTRACT

In this article, an existence theorem of global solutions with small initial data belonging to $L^1 \cap L^p$, $(n < p \leqslant \infty)$ for a chemotaxis system is given on the whole space $\mathbb{R}^n$, $n \geqslant 3$. In the case $p = \infty$, our global solution is integrable with respect to the space variable on some time interval, and then conserves the mass for a short time, at least. The system consists of a chemotaxis equation with a logarithmic term and an ordinary equation without diffusion term.

1. Introduction

This paper concerns global solutions to the system

$$(\mathrm{E}_\lambda) \quad \begin{cases} \partial_t u = \Delta u - \nabla \cdot \left(u \frac{\nabla v}{v} \right), & x \in \mathbb{R}^n,\ t > 0, \\ \partial_t v = u v^\lambda, \quad x \in \mathbb{R}^n,\ t > 0, \\ u(0,x) = a(x) \geqslant 0, \quad v(0,x) = b(x) \geqslant 0, \quad x \in \mathbb{R}^n. \end{cases}$$

Here, u is the unknown cell density of the chemotactic species and v is the unknown density of non-diffusive chemical substance, which is produced by the species. This system is a particular case of Keller–Segel system [8] and related to the dynamics of self-reinforced random walks [14,16], and also used as haptotaxis and angiogenesis models. One of interesting features of the system is the absence of diffusion term in the second equation. There are many papers which studied the classical Keller–Segel system with diffusion term in the second equation. Levine and Sleeman [10] investigated finite time blow-up phenomena for the system (E_1) in one dimensional case. Additional properties for the solutions of (E_1) have been obtained in [12]. In smooth bounded domains in $\mathbb{R}^n$ with $\lambda \leqslant 1$, Rascle [15] and Yang, Chen and Liu [20] showed the existence of global solutions for the system. Corrias, Perthame and Zaag [5,6] studied the same topics in a general system, which do not cover the case $\lambda = 1$. In [17], asymptotic behavior of radial symmetric solutions to (E_λ) is studied. When $\lambda \in [0,1)$ and $n = 1$, Kang, Stevens and Velázquez proved that for some initial data the corresponding solutions u tends to Dirac mass as $t \to \infty$ in [7].

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0022-247X/$ – see front matter
http://dx.doi.org/10.1016/j.jmaa.2013.08.065

Through the transformation

$$z = \frac{v^{1-\lambda}}{1-\lambda} \quad \text{with } \theta = \frac{1}{1-\lambda} \in \mathbb{R},$$

the system (E_λ) becomes

$$(\tilde{\mathrm{E}}_\theta) \quad \begin{cases} \partial_t u = \Delta u - \theta \nabla \cdot \left(u \frac{\nabla z}{z} \right), & x \in \mathbb{R}^n,\ t > 0, \\ \partial_t z = u, \quad x \in \mathbb{R}^n,\ t > 0, \\ u(0,x) = a(x), \quad z(0,x) = c(x) = \theta b(x)^{1-\lambda}, & x \in \mathbb{R}^n. \end{cases}$$

This system is invariant with respect to the scaling

$$\big(u(t,x), z(t,x)\big) \mapsto \big(\mu^{\alpha+2} u\big(\mu^2 t, \mu x\big), \mu^\alpha z\big(\mu^2 t, \mu x\big)\big)$$

for all $\alpha \in \mathbb{R}$. In this article, we give a global existence theorem of solutions to the system $(\tilde{\mathrm{E}}_\theta)$ with the special initial data $c \equiv 1$, which corresponds to the case $\lambda < 1$ in (E_λ), in the sense of mild solutions, more precisely, we construct solutions to the integral equations:

$$(\text{I.E.}) \quad u(t) = e^{t\Delta} a - \theta B[u](t), \qquad z(t) = 1 + \int_0^t u(\tau)\, d\tau,$$

where

$$B[u](t) = \int_0^t e^{(t-\tau)\Delta} \nabla \cdot \left(u(\tau) \frac{\nabla z(\tau)}{z(\tau)} \right) d\tau.$$

As a consequence of the absence of diffusion term in the second equation, the regularity of z and ∇z with respect to space variable is not better than that of 1 and ∇u respectively. For the initial data $c \equiv -1$ that corresponds to another case $\lambda > 1$ in (E_λ), the same results as in below hold. D. Li, K. Li and Zhao [11] treated with the case $\lambda = 1$ in which the system is changed to a hyperbolic–parabolic system through the transform $V = -\frac{\nabla v}{v}$, and constructed local and global solutions in Sobolev spaces with positive smoothness. Very recently, Ahn and Kang [1] proved the local and global existence of solutions to (E_λ) with some λ and θ, and also the non-existence of self-similar solutions.

Our main result reads as follows

Theorem 1.1 *(Small data global existence). Let $n \geqslant 3$, $n/(n-1) < q < n < r < p \leqslant \infty$ and $\theta \in \mathbb{R}$. There exists $\delta = \delta(n,p,q,r,|\theta|) > 0$ such that if $\|a\|_{L^1 \cap L^p} \leqslant \delta$, then there exists a global solution $u \in L^1(0,\infty; L^\infty(\mathbb{R}^n))$ of* (I.E.) *satisfying $\|u\|_{X^1_\infty} + \|u\|_{X^2_\infty} + \|u\|_{X^3_\infty} \lesssim \|a\|_{L^1 \cap L^p}$ where*

$$\|u\|_{X^1_t} = \int_0^t \big\|u(\tau)\big\|_{L^\infty}\, d\tau,$$

$$\|u\|_{X^2_t} = \sup_{\tau < t} \left\| \int_0^\tau \nabla u(\sigma)\, d\sigma \right\|_{L^r} \quad \text{and} \quad \|u\|_{X^3_t} = \sup_{\tau < t} \left\| \int_0^\tau \nabla u(\sigma)\, d\sigma \right\|_{L^q}.$$

Moreover, with $U_\infty(t) = \int_0^t \|u(\tau)\|_{L^\infty}\, d\tau$ and $U(t,x) = \int_0^t u(\tau,x)\, d\tau$, one has

$$U_\infty \in C\big([0,\infty)\big) \quad \text{and} \quad \nabla U \in C\big([0,\infty); L^r \cap L^q\big). \tag{1}$$

Remark 1.1. The smallness assumption on the initial data a is used to guarantee

$$\sup_{t>0} \left\| \frac{1}{z_0(t)} \right\|_{L^\infty} \lesssim 1, \quad \text{where } z_0(t) = 1 + \int_0^t e^{\tau\Delta} a\, d\tau.$$

Indeed, $\int_0^\infty \|e^{\tau\Delta} a\|_{L^\infty}\, d\tau \approx \|a\|_{\dot{B}^{-2}_{\infty,1}}$ and $L^1 \cap L^p \hookrightarrow \dot{B}^{-2}_{\infty,1}$. We then have

$$\big|z_0(t,x)\big| \geqslant 1 - \int_0^\infty \big|e^{\tau\Delta} a(x)\big|\, d\tau \geqslant 1/2.$$