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Inverse mean curvature flow with forced term ☆

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ABSTRACT

In this paper, we study the evolution of convex hypersurfaces by inverse mean curvature minus an external force field c. We prove that the flow will preserve the convexity for any c. If $c < \frac{1}{H}$ on the initial surface, we prove that the flow will expand the hypersurface for all time, and after rescaling the hypersurface will converge to a sphere. If $c > \frac{1}{H}$ on the initial surface, we show that the maximum existence time of the flow is finite, and the hypersurface will contract to a point when approaching the final time.

1. Introduction

In this paper, we study the flow

$$\frac{dF}{dt} = \left(\frac{1}{H} - c\right)\nu := f\nu \tag{1.1}$$

where

$$F_t := F(\cdot, t) : S^n \to R^{n+1}$$

is a family of closed hypersurfaces with $M_t = F_t(S^n)$. H denotes the mean curvature of M_t w.r.t. outer unit normal vector ν, and c is a constant.

For $c = 0$, this flow is called inverse mean curvature flow. Inverse mean curvature flow is an important method to derive the energy estimates in General Relativity, one can see Huisken and Ilmanen [8]. In [19], Urbas proved that for inverse mean curvature flow the surfaces stay strictly convex and smooth for all time. Furthermore, the surfaces become more and more spherical in the process. The similar results have also been obtained for star-shaped initial data with $H > 0$, see Urbas [18] and Gerhardt [3]. In [9], Huisken and Ilmanen proved a sharp lower bound of mean curvature, from which they proved that if the initial surface is strictly star-shaped and $H \geqslant 0$, the smooth solution of the inverse mean flow will exist for all time and converge to a manifold.

The curvature flows with forced term have been studied extensively. For mean curvature flow with external force field, which is a model describing the Ginzburg–Landau vortex, one can refer to [10,12–16]. There are also some other evolution equations for geometric quantities for general speeds, see e.g. [2,4].

In this paper, we want to find out how the force term c affects the flow. First we will prove that the flow preserve the convexity.

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Theorem 1.1. *Suppose the initial data M_0 of the flow (1.1) is strictly convex, then the strict convexity of the hypersurfaces is preserved during the evolution.*

For different c, we see that the flow (1.1) has different properties. The flow (1.1) may expand the surface if c is small enough (e.g. $c = 0$ [19]), or contract if c is large enough. The next theorem will tell us how the forced term c affects the flow.

Theorem 1.2. *Suppose M_0 is a strictly convex hypersurface. If $c < \frac{1}{H}$ on M_0, the flow (1.1) will expand the convex hypersurface for all time. And if we let $\widetilde{M}_t = e^{-\frac{1}{n}t} M_t$, $\widetilde{M}_t$ will converge in C^∞ topology to a sphere. If $c > \frac{1}{H}$ on M_0, the flow (1.1) will contract the convex hypersurface to a point in finite time.*

In our paper, strict convexity shall be defined as all the eigenvalues of the second fundamental form of the surface being positive. For the proof of Theorem 1.1, we will follow the method of Schulze in [17], and work directly with Eq. (1.1). To prove the expanding convex hypersurface will converge to a sphere after rescaling, we use the method of support function which is used as the main method in [19]. To prove that the convex hypersurface contracts to a point when $c > \frac{1}{H(0)}$, we mainly use the comparison principle.

This paper is organized as follows. In Section 2, we will give some notations and preliminaries. Moreover, we will give some evolution equations and obtain the short-time existence of the flow (1.1). Then in Section 3, we prove the flow preserving convexity by studying the evolution equation of the inverse of the second fundamental form. In Section 4, we prove that when $c > \frac{1}{H}$ on M_0, the hypersurface will contract to a point. In Section 5, we prove that if $c < \frac{1}{H}$ on M_0, the hypersurface will expand and finally complete the proof of Theorem 1.2.

2. Preliminaries

If M is given as in Section 1 and F denotes its parametrization in R^{n+1}, the metric $\{g_{ij}\}$ are given by

$$g_{ij}(x) = \left\langle \frac{\partial F(x)}{\partial x_i}, \frac{\partial F(x)}{\partial x_j} \right\rangle, \quad x \in M.$$

Let ∇ and Δ denote the Levi-Civita connection and the Laplace–Beltrami operator on M respectively. Indices are raised and lowered w.r.t. g^{ij} and g_{ij}. Einstein's notation is used for summation. We will use $\langle\, , \rangle$ to denote the standard scalar product on R^n.

The second fundamental form in direction ν is denoted by

$$h_{ij}(x) = -\langle \nu, \nabla_i \nabla_j F \rangle,$$

the norm of the second fundamental form is given by

$$|A|^2 = g^{ij} g^{pl} h_{ip} h_{lj},$$

and the mean curvature on M is given by

$$H = g^{ij} h_{ij}.$$

First, we have the following evolution equations for various geometric quantities under flow (1.1).

Proposition 2.1. *Suppose flow (1.1) holds true for $t \in [0, T)$ with $T \leqslant \infty$, then we have the following equations on M_t for all $t \in [0, T)$:*

$$\frac{dg_{ij}}{dt} = 2 f h_{ij}, \tag{2.1}$$

$$\frac{dg^{ij}}{dt} = -2 f h_{pl} g^{ip} g^{jl}, \tag{2.2}$$

$$\frac{dh_{ij}}{dt} = -\nabla_i \nabla_j f + f h_i^l h_{lj}, \tag{2.3}$$

$$\frac{dH}{dt} = -\Delta f - f|A|^2. \tag{2.4}$$

Proof. The evolution equations have been established in many papers, one can see [9,10,13] for details. □

Next, we will give the evolution of f.