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H^2 regularity for the $p(x)$-Laplacian in two-dimensional convex domains ☆

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ABSTRACT

In this paper we study the H^2 global regularity for solutions of the $p(x)$-Laplacian in two-dimensional convex domains with Dirichlet boundary conditions. Here $p : \Omega \to [p_1, \infty)$ with $p \in \mathrm{Lip}(\overline{\Omega})$ and $p_1 > 1$.

1. Introduction

Let Ω be a bounded domain in $\mathbb{R}^2$ and let $p : \Omega \to (1, +\infty)$ be a measurable function. In this work, we study the H^2 global regularity of the weak solution of the following problem

$$\begin{cases} -\Delta_{p(x)} u = f & \text{in } \Omega, \\ u = g & \text{on } \partial\Omega, \end{cases} \tag{1.1}$$

where $\Delta_{p(x)} u = \operatorname{div}(|\nabla u|^{p(x)-2}\nabla u)$ is the $p(x)$-Laplacian. The hypothesis over p, f and g will be specified later.

Note that, the $p(x)$-Laplacian extends the classical Laplacian ($p(x) \equiv 2$) and the p-Laplacian ($p(x) \equiv p$ with $1 < p < +\infty$). This operator has been recently used in image processing and in the modeling of electrorheological fluids, see [3,5,24].

Motivated by the applications to image processing problems, in [8], the authors study two numerical methods to approximate solutions of the type of (1.1). In Theorem 7.2, the authors prove the convergence in $W^{1,p(\cdot)}(\Omega)$ of the conformal Galerkin finite element method. It is of our interest to study, in a future work, the rate of this convergence. In general, all the error bounds depend on the global regularity of the second derivatives of the solutions, see for example [6,22]. However, there appear to be no existing regularity results in the literature that can be applied here, since all the results have either a first order or local character.

The H^2 global regularity for solutions of the p-Laplacian is studied in [22]. There the authors prove the following: Let $1 < p \leqslant 2$, $g \in H^2(\Omega)$, $f \in L^q(\Omega)$ ($q > 2$) and u be the unique weak solution of (1.1). Then:

- If $\partial\Omega \in C^2$ then $u \in H^2(\Omega)$;

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- If Ω is convex and $g = 0$ then $u \in H^2(\Omega)$;
- If Ω is convex with a polygonal boundary and $g \equiv 0$ then $u \in C^{1,\alpha}(\overline{\Omega})$ for some $\alpha \in (0, 1)$.

Regarding the regularity of the weak solution of (1.1) when $f = 0$, in [1,7], the authors prove the $C^{1,\alpha}_{loc}$ regularity (in the scalar case and also in the vectorial case). Then, in the paper [15] the authors study the case where the functional has the so-called (p, q)-growth conditions. Following these ideas, in [17], the author proves that the solutions of (1.1) are in $C^{1,\alpha}(\overline{\Omega})$ for some $\alpha > 0$ if Ω is a bounded domain in $\mathbb{R}^N$ $(N \geqslant 2)$ with $C^{1,\gamma}$ boundary, $p(x)$ is a Hölder function, $f \in L^\infty(\Omega)$ and $g \in C^{1,\gamma}(\overline{\Omega})$; while in [4], the authors prove that the solutions are in $H^2_{loc}(\{x \in \Omega\colon\ p(x) \leqslant 2\})$ if $p(x)$ is uniformly Lipschitz ($\mathrm{Lip}(\Omega)$) and $f \in W^{1,q(\cdot)}_{loc}(\Omega) \cap L^\infty(\Omega)$.

Our aim, it is to generalize the results of [22] in the case where $p(x)$ is a measurable function. To this end, we will need some hypothesis over the regularity of $p(x)$. Moreover, in all our result we can avoid the restriction $g = 0$, assuming some regularity of $g(x)$.

On the other hand, to prove our results, we can assume weaker conditions over the function f than the ones on [4]. Since, we only assume that $f \in L^{q(\cdot)}(\Omega)$, we do not have a priori that the solutions are in $C^{1,\alpha}(\Omega)$. Then we cannot use it to prove the H^2 global regularity. Nevertheless, we can prove that the solutions are in $C^{1,\alpha}(\overline{\Omega})$, after proving the H^2 global regularity.

The main results of this paper are:

Theorem 1.1. *Let Ω be a bounded domain in $\mathbb{R}^2$ with C^2 boundary, $p \in \mathrm{Lip}(\overline{\Omega})$ with $p(x) \geqslant p_1 > 1$, $g \in H^2(\Omega)$ and u be the weak solution of (1.1). If*

(F1) *$f \in L^{q(\cdot)}(\Omega)$ with $q(x) \geqslant q_1 > 2$ in the set $\{x \in \Omega\colon\ p(x) \leqslant 2\}$;*
(F2) *$f \equiv 0$ in the set $\{x \in \Omega\colon\ p(x) > 2\}$,*

then $u \in H^2(\Omega)$.

Theorem 1.2. *Let Ω be a bounded domain in $\mathbb{R}^2$ with convex boundary, $p \in \mathrm{Lip}(\overline{\Omega})$ with $p(x) \geqslant p_1 > 1$, $g \in H^2(\Omega)$ and u be the weak solution of (1.1). If f satisfies (F1) and (F2) then $u \in H^2(\Omega)$.*

Using the above theorem we can prove the following:

Corollary 1.3. *Let Ω be a bounded convex domain in $\mathbb{R}^2$ with polygonal boundary, p and f as in the previous theorem, $g \in W^{2,q(\cdot)}(\Omega)$ and u be the weak solution of (1.1) then $u \in C^{1,\alpha}(\overline{\Omega})$ for some $0 < \alpha < 1$.*

Observe that this result extends the one in [17] in the case where Ω is a polygonal domain in $\mathbb{R}^2$.

Organization of the paper. The rest of the paper is organized as follows. After a short Section 2 where we collect some preliminary results, in Section 3, we study the H^2-regularity for the non-degenerated problem. In Section 4 we prove Theorem 1.1. Then, in Section 5, we study the regularity of the solution u of (1.1) if Ω is convex. In Section 6, we make some comments on the dependence of the H^2-norm of u on p_1. Lastly, in Appendices A and B we give some results related to elliptic linear equation with bounded coefficients and Lipschitz functions, respectively.

2. Preliminaries

We now introduce the spaces $L^{p(\cdot)}(\Omega)$ and $W^{1,p(\cdot)}(\Omega)$ and state some of their properties.

Let Ω be a bounded open set of $\mathbb{R}^n$ and $p : \Omega \to [1, +\infty)$ be a measurable bounded function, called a variable exponent on Ω and denote $p_1 := \mathit{essinf}\, p(x)$ and $p_2 := \mathit{esssup}\, p(x)$.

We define the variable exponent Lebesgue space $L^{p(\cdot)}(\Omega)$ to consist of all measurable functions $u : \Omega \to \mathbb{R}$ for which the modular

$$\varrho_{p(\cdot)}(u) := \int_\Omega |u(x)|^{p(x)}\, dx$$

is finite. We define the Luxemburg norm on this space by

$$\|u\|_{L^{p(\cdot)}(\Omega)} := \inf\{k > 0\colon\ \varrho_{p(\cdot)}(u/k) \leqslant 1\}.$$

This norm makes $L^{p(\cdot)}(\Omega)$ a Banach space.

For the proofs of the following theorems, we refer the reader to [12].