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Multivalued fixed point theorems in terms of weak topology and measure of weak noncompactness[☆]

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ABSTRACT

We present several fixed point theorems for multimaps in Banach spaces, which in turn are multivalued versions of the Mönch, Daher, Darbo and Sadovskii theorems. The main assumptions of our results are formulated in terms of weak topology and an axiomatic definition of measure of weak noncompactness. Our theorems extend in a broad sense some new and classical results; moreover, a Sadovskii type theorem stated in 2011 is improved.

1. Introduction

The aim of the paper is to study the existence of fixed points in Banach spaces for multimaps assuming not necessarily compact values and having w-weakly closed graph, i.e. weakly closed graph with respect to the weak topology. We underline the interest that, in the last years, the fixed point theory under weak topology features holds in the literature (see, e.g. [2,3,6,12–14]).

The present paper is organized as follows.

After some preliminaries, in Section 3 we provide a fixed point theorem in separable Banach spaces (see Theorem 3.1) where we assume a Mönch type hypothesis on the multimap, formulated by means of the weak topology. In the norm topology setting, this property was introduced by O'Regan and Precup [15] in order to extend to multimaps the classical Mönch theorem.

In Section 4, we state a multivalued version of the Daher fixed point theorem (see Theorem 4.1). It is well known that the multimaps considered in this kind of theorems are countably condensing (see, e.g. [1,8]). To achieve our goal, we give this notion by means of a measure of noncompactness which "reads" the weak topology. The first measure of weak noncompactness was defined by De Blasi [9] as the counterpart for the weak topology of the classical Hausdorff measure of noncompactness. Being sometimes very difficult to use the De Blasi measure of weak noncompactness in the applications, Xu and Ben Amar [17] gave an axiomatic approach to measures of weak noncompactness starting from the paper of Banaś and Rivero [5]. Actually, some of the properties listed in their definition can be deduced from the others. In the present paper, we propose a cleaner axiomatic definition of measure of weak noncompactness (see Definition 4.1), which is on the line to the Appell's survey on measures of noncompactness [4].

Further, from our Daher type theorem we deduce a Darbo type fixed point principle for multimaps.

In Section 5, we give a Sadovskii type theorem in Banach spaces which are not necessarily separable (see Theorem 5.1).

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At the end of the paper we draw some inferences from our precedent results. First, we note that the results in Sections 3 and 4 extend in a broad sense the analogous theorems presented in [8] (and therefore also those in [1,15]) where the multimaps take compact values. Finally, it is worth pointing out that our Theorem 5.1 improves the classical fixed point theorem for condensing multimaps (see [11, Corollary 3.3.1]) and also a recent Sadovskii type result due to Agarwal, O'Regan, Taoudi (cf. [3, Theorem 2.1]).

2. Preliminaries

Let $\mathcal{X}$ be a Hausdorff topological linear space and $\mathcal{P}(\mathcal{X})$ be the family of all nonempty subsets of $\mathcal{X}$. Following [7], we recall the next notion.

Definition 2.1. Let D be a nonempty subset of $\mathcal{X}$. A map $F : D \to \mathcal{P}(\mathcal{X})$ is said to have *weakly closed graph* in $D \times \mathcal{X}$ if for every net $(x_\delta)_\delta$ in D, $x_\delta \to x$, $x \in D$, and for every net $(y_\delta)_\delta$, $y_\delta \in F(x_\delta)$, $y_\delta \to y$, we have $S(x, y) \cap F(x) \neq \emptyset$, where $S(x, y) = \{x + \lambda(y - x) : \lambda \in [0, 1]\}$.

In the sequel we will use the following theorem.

Theorem 2.1 ([7, Theorem III])**.** *Let $\mathcal{X}$ be a Hausdorff locally convex topological linear space, D be a nonempty subset of $\mathcal{X}$ and $F : D \to \mathcal{P}(\mathcal{X})$ be a map with the properties*

(α) *$F(x)$ is closed and convex, for every $x \in D$;*
($\alpha\alpha$) *F has weakly closed graph in $D \times \mathcal{X}$.*

Under these conditions, the multimap F has a fixed point if and only if there exists a compact and convex set $K \subset D$ satisfying $F(x) \cap K \neq \emptyset$, for every $x \in K$.

3. Mönch type multimaps

Let X be a Banach space and $\mathcal{T}_w$ be the weak topology on X. Fixed $D \in \mathcal{P}(X)$, we say that a map $F : D \to \mathcal{P}(X)$ has *w-weakly closed graph* in $D \times X$ if it has weakly closed graph in $D \times X$ with respect to the weak topology.

Let $\mathcal{P}_{fc}(D)$ be the family of all nonempty closed convex subsets of D and $\overline{D}^w$ be the weak closure of D. If $\overline{D}^w$ is weakly compact (w-compact, for short), the set D is said to be relatively w-compact.

Our first result provides the existence of fixed points for Mönch type multimaps taking not necessarily compact values (unlike [1,8,15]).

Theorem 3.1. *Let D be a closed convex subset of a separable Banach space X and $F : D \to \mathcal{P}_{fc}(D)$ be a map such that*

(i) *F has w-weakly closed graph in $D \times X$;*
(ii) *F maps w-compact sets into relatively w-compact sets.*

Suppose also that the following property is satisfied:

(wM) *there exists $x_0 \in D$ such that*

$$\left.\begin{array}{r} M \subset D, M = co(\{x_0\} \cup F(M)) \\ \text{and } \overline{M}^w = \overline{C}^w \text{ with } C \subset M \text{ countable} \end{array}\right\} \Rightarrow \overline{M}^w \text{ is } w\text{-compact}.$$

Then there exists $x \in D$ with $x \in F(x)$.

Proof. Let $x_0 \in D$ as by (wM).

We consider the iterative sequence $(M_n)_{n\in\mathbb{N}}$ of sets:

$$M_0 = \{x_0\}; \qquad M_n = co(\{x_0\} \cup F(M_{n-1})), \quad n \in \mathbb{N}^+.$$

Being M_n and D convex sets and D also closed, we have

$$\overline{M_n}^w = \overline{M_n} \subset D, \quad n \in \mathbb{N}. \tag{1}$$

Further, it is easy to see by induction that

$$M_{n-1} \subset M_n, \quad n \in \mathbb{N}^+. \tag{2}$$

Now, M_0 is clearly w-compact. Let us prove by induction that for every $n \in \mathbb{N}^+$ the set M_n is relatively w-compact. First, by hypothesis (ii) the set $\{x_0\} \cup \overline{F(M_0)}^w$ is w-compact. Therefore, since $\overline{M_1}^w \subset \overline{co}(\{x_0\} \cup \overline{F(M_0)}^w)$, by applying the Krein–Smulian Theorem (see, e.g. [10, Theorem 3.5.15]) we can deduce that M_1 is relatively w-compact. Now, suppose that M_{n-1} is relatively