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Logarithmically improved blow-up criteria for a phase field Navier–Stokes vesicle–fluid interaction model[☆]

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ABSTRACT

In this paper, we study a hydrodynamical system modeling the deformation of the vesicle membrane under external incompressible viscous flow fields. The system is in the Eulerian formulation and is governed by the coupling of the incompressible Navier–Stokes equations with a phase field equation. In the three dimensional case, we establish two logarithmically improved blow-up criteria for local smooth solutions of this system in terms of the vorticity field only in the homogeneous Besov spaces.

1. Introduction

Recently, there have been many numerical and theoretical studies on the configurations and deformations of elastic vesicle membranes under external flow fields [3,4,6–9,17,19,24,25]. The single component vesicle membranes are possibly the simplest models for the biological cells and molecules and have widely studied in biology, biophysics and bioengineering. Such vesicle membranes can be formed by certain amphiphilic molecules assembled in water to build bilayers, and exhibit a rich set of geometric structures in various mechanical, physical and biological environments [7]. In order to model and understand the formation and dynamics of vesicle membranes and the fluid–structure interaction, one approach is to consider equations of elasticity for the vesicle membranes and the Navier–Stokes equations for the fluid. However, the model established in this approach is very difficult to study and numerically simulate due to the fact that the description for elasticity is in the Lagrangian coordinate (Hooke's law) and for fluids is in the Eulerian coordinate. To overcome this difficulty, in [4,7], the authors established a phase field Navier–Stokes vesicle–fluid interaction model for the vesicle shape dynamics in flow fields via the phase field approach. In this model, the vesicle membrane Γ is described by a phase function ϕ, which is a labeling function defined on computational domain Q. The function ϕ takes value $+1$ inside the vesicle membrane and -1 outside, with a thin transition layer of width characterized by a small (compared to the vesicle size) positive parameter ε. Obviously, the sharp transition layer of the phase function gives a diffusive interface description of the vesicle membrane Γ, which is recovered by the zero level set $\{x : \phi(x) = 0\}$. The advantage of introducing such a phase function ϕ is to formulate the original Lagrangian description of the membrane evolution in the Eulerian coordinates. On the other hand, the viscous fluid is modeled by the incompressible Navier–Stokes equations with unit density and with an external force defined in terms of ϕ.

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In this paper, we study the three dimensional phase field Navier–Stokes vesicle–fluid interaction model subjecting to the periodic boundary conditions (i.e., in torus $\mathbb{T}^3$), which reads as follows:

$$\partial_t u + u \cdot \nabla u + \nabla P = \mu \Delta u + \frac{\delta E(\phi)}{\delta \phi} \nabla \phi \quad \text{in } Q \times [0, T], \tag{1.1}$$

$$\nabla \cdot u = 0 \quad \text{in } Q \times [0, T], \tag{1.2}$$

$$\partial_t \phi + u \cdot \nabla \phi = -\gamma \frac{\delta E(\phi)}{\delta \phi} \quad \text{in } Q \times [0, T] \tag{1.3}$$

with the initial conditions

$$u(x, 0) = u_0(x) \quad \text{with } \nabla \cdot u_0 = 0, \quad \text{and} \quad \phi(x, 0) = \phi_0(x) \quad \text{for } x \in Q, \tag{1.4}$$

and the boundary conditions

$$u(x + e_i, t) = u(x, t), \qquad \phi(x + e_i, t) = \phi(x, t) \quad \text{for } x \in \partial Q \times [0, T],\ i = 1, 2, 3, \tag{1.5}$$

where the set of vectors $\{e_1 = (1, 0, 0), e_2 = (0, 1, 0), e_3 = (0, 0, 1)\}$ denotes an orthonormal basis of $\mathbb{R}^3$ and Q is the unit square in $\mathbb{R}^3$. Here $u = (u_1, u_2, u_3) \in \mathbb{R}^3$ and $P \in \mathbb{R}$ denote the unknown velocity vector field and the unknown scalar pressure of the fluid, respectively. $\phi \in \mathbb{R}$ is the phase function of the vesicle membrane Γ. $E(\phi)$ denotes the physical approximation/regularization of the Helfrich elastic bending energy for the vesicle membrane which is given by (cf. [4,6,8,9])

$$E(\phi) = E_\varepsilon(\phi) + \frac{1}{2} M_1 (A(\phi) - \alpha)^2 + \frac{1}{2} M_2 (B(\phi) - \beta)^2 \tag{1.6}$$

with

$$E_\varepsilon(\phi) = \frac{k}{2\varepsilon} \int_Q |f(\phi)|^2 dx \quad \text{and} \quad f(\phi) = -\varepsilon \Delta \phi + \frac{1}{\varepsilon}(\phi^2 - 1)\phi, \tag{1.7}$$

where ε is a small positive parameter that characterizes the thickness of the transition layer of the phase function, M_1 and M_2 are two penalty constants which are introduced to enforce the volume

$$A(\phi) = \int_Q \phi \, dx \tag{1.8}$$

and the surface area

$$B(\phi) = \int_Q \left(\frac{\varepsilon}{2} |\nabla \phi|^2 + \frac{1}{4\varepsilon} (\phi^2 - 1)^2 \right) dx \tag{1.9}$$

of the vesicle conserved (in time), and $\alpha = A(\phi_0)$ and $\beta = B(\phi_0)$ are determined by the initial value of the phase function ϕ_0. The positive constants ν, k, and γ denote, respectively, the viscosity of the fluid, the bending modulus of the vesicle, and the mobility coefficient. $\frac{\delta E(\phi)}{\delta \phi}$ is the so-called chemical potential that denotes the variational derivative of $E(\phi)$ in the variable ϕ. Note that, if we denote

$$g(\phi) = -\Delta f(\phi) + \frac{1}{\varepsilon^2}(3\phi^2 - 1) f(\phi), \tag{1.10}$$

then a direct calculation yields that the variation of the approximate elastic energy is given by (see [4,6])

$$\begin{aligned} \frac{\delta E(\phi)}{\delta \phi} &= k g(\phi) + M_1 (A(\phi) - \alpha) + M_2 (B(\phi) - \beta) f(\phi) \\ &= k\varepsilon \Delta^2 \phi - \frac{k}{\varepsilon} \Delta(\phi^3 - \phi) - \frac{k}{\varepsilon}(3\phi^2 - 1)\Delta\phi + \frac{k}{\varepsilon^3}(3\phi^2 - 1)(\phi^2 - 1)\phi \\ &\quad + M_1 (A(\phi) - \alpha) + M_2 (B(\phi) - \beta) f(\phi). \end{aligned} \tag{1.11}$$

The system (1.1)–(1.3) describes the dynamic evolution of vesicle membranes immersed in an incompressible, Newtonian fluid, using an energetic variational approach [4,7] (see [5,6,9,21,24] for numerical simulations and other studies). In 2007, Du, Li and Liu in [4] studied well-posedness of the system (1.1)–(1.3) subjecting to the no-slip boundary condition for the velocity field and the Dirichlet boundary condition for the phase function, by using the modified Galerkin argument, the authors established the global existence of weak solutions and the basic energy inequality

$$\frac{d}{dt} \left(\frac{1}{2} \|u(\cdot, t)\|_{L^2}^2 + E(\phi(\cdot, t)) \right) + \mu \|\nabla u(\cdot, t)\|_{L^2}^2 + \gamma \left\| \frac{\delta E(\phi)}{\delta \phi} \right\|_{L^2}^2 = 0, \quad \forall t > 0. \tag{1.12}$$

Moreover, the authors also proved that the weak solution is unique under an additional regularity assumption $u \in L^8(0, T; L^4(Q))$. Recently, local in time existence and uniqueness of the strong solution to the system (1.1)–(1.3) have been established in [19], and under the assumptions that the initial data and the quantity $(|Q| + \alpha)^2$ are sufficiently small, the