J. Math. Anal. Appl. 405 (2013) 555–564

Contents lists available at SciVerse ScienceDirect

Journal of Mathematical Analysis and Applications

journal homepage: www.elsevier.com/locate/jmaa

On the fundamental solution for higher spin Dirac operators

D. Eelbode [a], T. Raeymaekers [b,*], P. Van Lancker [b]

[a] Department of Mathematics and Computer Science, University of Antwerp, Campus Middelheim, G-Building, Middelheimlaan 1, 2020 Antwerpen, Belgium

[b] Clifford Research Group, Department of Mathematical Analysis, Ghent University, Galglaan 2, 9000 Ghent, Belgium

ARTICLE INFO

Article history:
Received 5 December 2012
Available online 16 April 2013
Submitted by Eero Saksman

Keywords:
Clifford analysis
Fundamental solution
Higher spin
Dirac operator
The Stokes theorem
The Cauchy integral formula

ABSTRACT

In this paper, we will determine the fundamental solution for the higher spin Dirac operator $\mathcal{Q}_\lambda$, which is a generalisation of the classical Rarita–Schwinger operator to more complicated irreducible (half-integer) representations for the spin group in m dimensions. This will allow us to generalise the Stokes theorem, the Cauchy–Pompeiu theorem and the Cauchy integral formula, which lie at the very heart of the function theory behind arbitrary elliptic higher spin operators.

1. Introduction

This article is to be situated in the theory of Clifford analysis, a generalisation of classical complex analysis in the plane to the case of an arbitrary dimension $m \in \mathbb{Z}$ (in the case of a negative dimension, one is dealing with so-called super-Clifford analysis). At the heart of the theory lies the Dirac operator ∂_x on $\mathbb{R}^m$, a conformally invariant first-order differential operator which plays the same role in classical Clifford analysis as the Cauchy–Riemann operator ∂_z does in complex analysis. Moreover, the Dirac operator satisfies the relation $\partial_x^2 = -\Delta_x$, which means that Clifford analysis can be seen as a refinement of harmonic analysis on $\mathbb{R}^m$.

The classical theory is centred around the study of functions on $\mathbb{R}^m$ which take values in the complex Clifford algebra $\mathbb{C}_m$ or a corresponding Spin(m)-subrepresentation, known as the spinor spaces (cf. [1,6,10]). In recent years, several authors [2–4,8] have been studying generalisations of classical Clifford analysis techniques to the so-called higher spin theory. This brings us to higher spin Dirac operators (or HSD-operators for short): generalised Dirac operators acting on functions on $\mathbb{R}^m$, which take values in arbitrary irreducible representations $\mathcal{S}_\lambda^\pm$ of the Spin(m)-group, with dominant half-integer highest weights. An explicit expression for these HSD-operators, which can be seen as generalised gradients in the sense of Stein and Weiss, was determined in [7]. The first generalisation appearing in Clifford analysis was the Rarita–Schwinger operator, originally inspired by equations coming from theoretical physics (see [12]). In the present context it is considered as the conformally invariant operator acting on functions taking values in $\mathcal{S}_1^\pm$ (see below for a definition).

In classical Clifford analysis, the Cauchy integral formula has proved to be a cornerstone of the function theory: it can be used to decompose arbitrary null solutions for the Dirac operator into homogeneous components and forms the basis for developing boundary value theory. This article explains how a higher spin version of this formula can be obtained. Cauchy integral formulae naturally rely upon the existence of a fundamental solution for the (higher spin) Dirac operator.

* Corresponding author.
E-mail addresses: David.Eelbode@ua.ac.be (D. Eelbode), tr@cage.ugent.be (T. Raeymaekers), Peter.VanLancker@hogent.be (P. Van Lancker).

0022-247X/$ – see front matter
http://dx.doi.org/10.1016/j.jmaa.2013.04.030

That is why, in the first place, a fundamental solution for $\mathcal{Q}_\lambda$ will be constructed, here relying on results from distribution theory. Also, this will lead to a generalised Stokes theorem and a Cauchy–Pompeiu theorem.

2. On Clifford analysis

The universal Clifford algebra $\mathbb{R}_m$ is the associative algebra generated by an orthonormal basis $(e_1, \ldots, e_m)$ for $\mathbb{R}^m$. The multiplication in the Clifford algebra is governed by the relations $e_ie_j + e_je_i = -2\delta_{ij}$, for all $i, j \in \{1, \ldots, m\}$. The complexification of $\mathbb{R}_m$ is defined by $\mathbb{C}_m = \mathbb{R}_m \otimes \mathbb{C}$. Let e_A be a basis element of $\mathbb{C}_m$, defined as $e_A = e_{i_1} \cdot e_{i_2} \cdot \ldots \cdot e_{i_h}$, with $i_1 < i_2 < \cdots < i_h$. The reversion or main anti-involution $a \mapsto a^*$ is defined on basis elements by means of $e_A^* = e_{i_h} \cdot \ldots \cdot e_{i_1}$, and is then linearly extended to the entire Clifford algebra $\mathbb{C}_m$:

$$(a_Ae_A + a_Be_B)^* = a_Ae_A^* + a_Be_B^*,$$

for all $a_A, a_B \in \mathbb{C}$. This anti-involution has the property that $(ab)^* = b^*a^*$ for all $a, b \in \mathbb{C}_m$. Analogously, we can define the Hermitean conjugation $a \mapsto a^\dagger$. On the basis elements e_A, it is defined by means of $e_A^\dagger = (-1)^h e_A^* = (-1)^h e_{i_h} \cdot \ldots \cdot e_{i_1}$, and it is then extended anti-linearly:

$$(a_Ae_A + a_Be_B)^\dagger = \bar{a}_Ae_A^\dagger + \bar{a}_Be_B^\dagger,$$

for all $a_A, a_B \in \mathbb{C}$. Here, $\bar{\cdot}$ denotes the complex conjugation. An alternative basis for $\mathbb{C}_m$, which will turn out to be very convenient for introducing the spinor spaces, is the so-called Witt basis:

Definition 1. The Witt basis in $\mathbb{C}_{2n}$ is defined by means of

$$\left(\mathfrak{f}_j, \mathfrak{f}_j^\dagger\right) := \left(\frac{e_j - ie_{j+n}}{2}, -\frac{e_j + ie_{j+n}}{2}\right),$$

where $1 \le j \le n$.

This basis has the properties that $\mathfrak{f}_j\mathfrak{f}_k = -\mathfrak{f}_k\mathfrak{f}_j$, $\mathfrak{f}_j^\dagger\mathfrak{f}_k^\dagger = -\mathfrak{f}_k^\dagger\mathfrak{f}_j^\dagger$ and $\mathfrak{f}_j\mathfrak{f}_k^\dagger + \mathfrak{f}_k^\dagger\mathfrak{f}_j = \delta_{jk}$. In terms of these basis elements, one can then define the idempotent $I = \mathfrak{f}_1\mathfrak{f}_1^\dagger \cdots \mathfrak{f}_n\mathfrak{f}_n^\dagger$, satisfying $I^2 = I$. We will now introduce the spinor space(s) $\mathbb{S}_{2n}^\pm$, vector spaces carrying the basic half-integer representations for the Spin(m)-group, which can be realised inside the Clifford algebra $\mathbb{C}_m$ by means of

$$\mathrm{Spin}(m) := \left\{ s = \prod_{j=1}^{2k} \omega_j : k \in \mathbb{N},\ \omega_j \in S^{m-1} \right\},$$

where S^{m-1} denotes the unit sphere in $\mathbb{R}^m$. Note that the parity sign will only play a role in the case where the dimension $m = 2n$ is even. First of all, we define the complex vector space $\mathbb{S}_{2n} = \mathbb{C}_{2n}I$. This space carries a canonical multiplicative action of the Clifford algebra $\mathbb{C}_{2n}$, denoted by $\gamma : \mathbb{C}_{2n} \to \mathrm{End}(\mathbb{S}_{2n})$ and defined by means of $\gamma(a)[\psi] = a\psi$, for all $a \in \mathbb{C}_{2n}$ and $\psi \in \mathbb{S}_{2n}$. As Spin$(2n) \subset \mathbb{C}_{2n}$, we can now also restrict this representation for $\mathbb{C}_{2n}$ to the spin group. However, as each element of Spin$(2n)$ belongs to the *even* subalgebra $\mathbb{C}_{2n}^+$ of $\mathbb{C}_{2n}$, the restriction of γ to Spin$(2n)$ splits into two irreducible subrepresentations (the so-called spinor representations), given by

$$\rho_\pm : \mathrm{Spin}(2n) \to \mathrm{Aut}(\mathbb{S}_{2n}^\pm),$$

where $\mathbb{S}_{2n} = \mathbb{S}_{2n}^+ \oplus \mathbb{S}_{2n}^-$, and $\mathbb{S}^{\pm_{2n}}$ are the graded subspaces of $\mathbb{S}_{2n}$. The highest weights for these representations are the basic half-integer highest weights $(\frac{1}{2}, \ldots, \frac{1}{2}, \pm\frac{1}{2})$. In the case where the dimension $m = 2n+1$ is odd, there exists a *unique* spinor representation for Spin(m), which amounts to saying that the parity index can be omitted. To define this representation, we first note that Spin$(2n+1) \subset \mathbb{C}_{2n+1}^+ \cong \mathbb{C}_{2n}$, where the algebra isomorphism can be defined in terms of the basis vectors by means of

$$\psi : \mathbb{C}_{2n} \to \mathbb{C}_{2n+1}^+ : e_j \mapsto e_je_m \quad (1 \le j \le 2n).$$

Using this isomorphism, we can define the spinor representation for Spin$(2n + 1)$ as follows:

$$\rho : \mathrm{Spin}(2n+1) \to \mathrm{Aut}(\mathbb{S}_{2n}) : s \mapsto \gamma\big(\psi^{-1}(s)\big).$$

Another option is to define the action of Spin$(2n + 1)$ on $\mathbb{S}_{2n+2}^\pm$, taking into account that both spinor spaces then become *isomorphic*. Note that we will from now on omit the dimension in the notation for spinor spaces, and write $\mathbb{S}^\pm$ instead of $\mathbb{S}^{\pm_{2n}}$. The parity index should then be omitted in the case where $m = 2n + 1$ is odd.

As was mentioned in the introduction, the Dirac operator is a key operator in Clifford analysis (cf. [1,6,10]). It is defined by means of

$$\partial_x := \sum_{j=1}^m e_j\partial_{x_j} \in \mathrm{Hom}_{\mathbb{C}}\big(\mathcal{C}^\infty(\mathbb{R}^m, \mathbb{S}^\pm), \mathcal{C}^\infty(\mathbb{R}^m, \mathbb{S}^\mp)\big).$$