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Normal families of functions for subelliptic operators and the theorems of Montel and Koebe

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ABSTRACT

A classical theorem of Montel states that a family of holomorphic functions on a domain $\Omega \subseteq \mathbb{C}$, uniformly bounded on the compact subsets of Ω, is a normal family. The aim of this paper is to obtain a generalization of this result in the subelliptic setting of families of solutions u to $\mathcal{L}u = 0$, where $\mathcal{L}$ belongs to a wide class of real divergence-form PDOs, comprising sub-Laplacians on Carnot groups, subelliptic Laplacians on arbitrary Lie groups, as well as the Laplace–Beltrami operator on Riemannian manifolds. To this end, we extend another remarkable result, due to Koebe: we characterize the solutions to $\mathcal{L}u = 0$ as fixed points of suitable mean-value operators with non-trivial kernels. A suitable substitute for the Cauchy integral formula is also provided. Finally, the local-boundedness assumption is relaxed, by replacing it with L^1_{loc}-boundedness.

1. Introduction and main results

The aim of this paper is to extend a celebrated theorem by P. Montel on normal families of holomorphic functions to the subelliptic setting of families of solutions u to $\mathcal{L}u = 0$, where $\mathcal{L}$ belongs to a wide class of real second-order divergence-form PDOs, comprising sub-Laplacians on Carnot groups, subelliptic Laplacians on arbitrary Lie groups, elliptic operators in divergence form, as well as the Laplace–Beltrami operator on Riemannian manifolds. To this aim, we provide a subelliptic version of another remarkable result, due to P. Koebe, characterizing harmonic functions as fixed points of suitable mean-value integral operators. The presence of non-trivial and possibly unbounded kernels in these mean-value operators is one of the main novelties with respect to the classical elliptic case.

Our investigation is motivated by the renewed interest in normal families in various contexts (see e.g., the recent monographs by Chuang [17] and by Schiff [45], and the survey papers [53,54] by Zalcman), this interest being further strengthened by the major rôle of normal families in Complex Dynamics. As anticipated, our study provides a contribution to the investigation of normal families of functions related to *subelliptic* operators, a setting in which, to the best of our knowledge, normal families have not been studied yet (see the bibliographical notes at the end of this introduction). We remark that we do *not* need to require any hypoellipticity assumption on the operators $\mathcal{L}$ involved here, nor any potential-theoretic regularity assumption on the sheaf of $\mathcal{L}$-harmonic functions.

Among the normality theorems usually named after Montel, [40], we are interested in the following one, concerning *locally bounded* families (see e.g., [31, Theorem 4, p. 80]): let $\mathcal{F}$ be a family of holomorphic functions on a domain $\Omega \subseteq \mathbb{C}$, uniformly bounded on the compact subsets of Ω; then $\mathcal{F}$ is a normal family, that is, given any compact set $K \subset \Omega$, every sequence in $\mathcal{F}$ admits a subsequence which is uniformly convergent on K.

As it is well known, there is a more refined normality theorem due to Montel, ensuring normality for a family of meromorphic functions (defined on a domain of the complex plane and valued in the extended complex plane $\mathbb{C} \cup \{\infty\} \simeq \mathbb{P}^1(\mathbb{C})$)

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omitting a triple of different values in $\mathbb{C} \cup \{\infty\}$. This result is deeply entrenched with complex function theory (for instance, it easily proves Picard's Great Theorem), and its generalization to the wide class of real PDOs we are considering here is out of our investigation. It is, instead, Montel's normality criterion on locally-bounded families that seems more suitable to be extended to other contexts; for example, this was obtained in [13,28] for holomorphic mappings in Banach spaces.

In the present investigation we point out some non-trivial tools required for extending to our context the previously mentioned normality theorem for locally-bounded families, such as the use of a Koebe-type result (characterizing the solutions to $\mathcal{L}u = 0$ as fixed points of suitable integral operators; see Theorem 1.2), and, mostly, an appropriate substitute for the Cauchy integral formula (the latter playing a key rôle in the proof of Montel's theorem; see Lemma 1.1). As byproducts, we shall obtain results on the topology of the $\mathcal{L}$-harmonic functions (see Lemma 1.5 and Theorem 1.6).

We now describe more closely the results of this paper. Throughout, we shall be concerned with linear second order PDOs in $\mathbb{R}^N$ of the form

$$\mathcal{L} := \frac{1}{V} \sum_{i,j=1}^{N} \partial_{x_i} (V\, a_{i,j}\, \partial_{x_j}), \tag{1}$$

where V is a C^1 positive function on $\mathbb{R}^N$, the matrix $A(x) := (a_{i,j}(x))_{i,j\le N}$ is symmetric and positive *semi*-definite at every point $x \in \mathbb{R}^N$, and it has C^1 entries. We assume that $\mathcal{L}$ is endowed with a positive fundamental solution $\Gamma(x, y)$, defined out of the diagonal of $\mathbb{R}^N \times \mathbb{R}^N$, with some well-behaved properties (see Section 2 for the precise requirements); see also Remark 2.1 for notable examples of operators as in (1) to which our theory applies (mainly, subelliptic operators on Lie groups, hence, in particular, sub-Laplacians on Carnot groups; see [11]).

If $\Omega \subseteq \mathbb{R}^N$ is open, we say that u is *$\mathcal{L}$-harmonic* in Ω if $u \in C^2(\Omega, \mathbb{R})$ and $\mathcal{L}u = 0$ in Ω. The set of the $\mathcal{L}$-harmonic functions in Ω will be denoted by $\mathcal{H}(\Omega)$. Given any $r > 0$ and any $x \in \mathbb{R}^N$, we introduce the super-level set of Γ

$$\Omega_r(x) := \left\{y \in \mathbb{R}^N : \Gamma(x, y) > 1/r\right\} \cup \{x\}, \tag{2}$$

that will be referred to as the *Γ-ball of center x and radius r*.

Given $\alpha > 0$, if H^α is the α-dimensional Hausdorff measure on $\mathbb{R}^N$, we set

$$\mathrm{d}V^\alpha := V\, \mathrm{d}H^\alpha \tag{3}$$

to denote the absolutely continuous measure with respect to H^α with density V. We shall be interested only in the cases $\alpha = N$ and $\alpha = N - 1$.

Let $x \in \mathbb{R}^N$ and let us consider the functions, defined for $y \neq x$,

$$\Gamma_x(y) := \Gamma(x, y), \qquad K(x, y) := \frac{\langle A(y)\nabla\Gamma_x(y), \nabla\Gamma_x(y)\rangle}{|\nabla\Gamma_x(y)|}. \tag{4}$$

If u is a continuous function on $\partial\Omega_r(x)$, we introduce the following mean-value operator

$$m_r(u)(x) := \int_{\partial\Omega_r(x)} u(y)\, K(x, y)\, \mathrm{d}V^{N-1}(y). \tag{5}$$

Note that the measure $K(x, y)\, \mathrm{d}V^{N-1}(y)$ is non-negative since A is positive semi-definite (we shall also prove that $\partial\Omega_r(x)$ has measure 1 w.r.t. $K(x, y)\, \mathrm{d}V^{N-1}(y)$). Mean-value operators such as (5) provide very versatile tools in the study of subelliptic PDOs: for instance, they are used to characterize the $\mathcal{L}$-subharmonic functions, [7,10]; they can be employed in studying the Dirichlet problem related to $\mathcal{L}$ and inverse mean-value theorems, [1]; in the special case of Carnot groups, where a homogeneous structure is available, they can also be used to derive invariant Harnack inequalities and Liouville-type theorems, [6], as well as Bôcher-type theorems for the removal of singularities, [8]. As a goal of this paper we show that they are useful tools also in the study of normal families in $\mathcal{H}(\Omega)$.

Our first result in the development of a convenient theory of normal families related to $\mathcal{L}$ is the following representation formula:

Lemma 1.1. *Let notation be as above. For every function u of class C^2 on an open set containing the Γ-ball $\overline{\Omega_r(x)}$, we have*

$$u(x) = m_r(u)(x) - \int_{\Omega_r(x)} \left(\Gamma(x, y) - \frac{1}{r}\right) \mathcal{L}u(y)\, \mathrm{d}V^N(y). \tag{6}$$

We shall refer to (6) as the surface mean-value formula for $\mathcal{L}$. *As a consequence, a function u of class C^2 in the open set $\Omega \subseteq \mathbb{R}^N$ is $\mathcal{L}$-harmonic if and only if*

$$u(x) = m_r(u)(x), \quad \text{for every } \Gamma\text{-ball such that } \overline{\Omega_r(x)} \subset \Omega. \tag{7}$$

We shall prove Lemma 1.1 by exploiting the quasi-divergence form (1) of $\mathcal{L}$ and integration by parts. Formula (6) extends the result in [10, Theorem 3.3] to our operators (1), a class which strictly contains the PDOs considered in [10].