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Absolute flux optimising curves of flows on a surface

Sanjeeva Balasuriya [a,b,*], Gary Froyland [c], Naratip Santitissadeekorn [d]

[a] School of Mathematical Sciences, University of Adelaide, Adelaide SA 5005, Australia
[b] Department of Mathematics, Connecticut College, New London CT 06320, USA
[c] School of Mathematics and Statistics, University of New South Wales, Sydney NSW 2052, Australia
[d] Department of Mathematics, University of North Carolina, Chapel Hill NC 27599, USA

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ABSTRACT

Given a flow on a surface, we consider the problem of connecting two distinct trajectories by a curve of extremal (absolute) instantaneous flux. We develop a complete classification of flux optimal curves, accounting for the possibility of the flux having spatially and temporally varying weight. This weight enables modelling the flux of non-equilibrium distributions of tracer particles, pollution concentrations, or active scalar fields such as vorticity. Our results are applicable to all smooth autonomous flows, area preserving or not.

1. Introduction

Given a two-dimensional manifold Ω, possibly with nonempty boundary, and a C^2 function $\mathbf{f} : \Omega \to \mathbb{R}^2$ we consider the dynamical system

$$\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}) \tag{1}$$

in which $\mathbf{x} \in \Omega$. We will search for flux extremising curves that join a specified pair of points $\mathbf{a}$ and $\mathbf{b}$. The "flux" here will be a general quantity; our definitions will provide for the quantification of each of a variety of entities crossing the curve per unit time, including the quantity of fluid, heat or chemical, or the amount of vorticity or potential vorticity. Clearly if $\mathbf{a}$ and $\mathbf{b}$ lie on a single trajectory of the flow, one may travel along this trajectory from $\mathbf{a}$ to $\mathbf{b}$ and incur zero flux; this trajectory segment would be the flux minimising curve. To pose nontrivial questions about flux extremising curves, we need to restrict our attention to pairs of points than cannot be joined by trajectories. This leads us to the notion of an *integral set*. Let $\phi(\mathbf{x}, t)$ be the flow which is generated by (1); that is, $\phi(\mathbf{x}, t)$ is the location in Ω to which an initial condition $\mathbf{x}$ progresses by time t.

Definition 1 (*Closed Trajectory*)**.** For any $\mathbf{x} \in \Omega$, define its *closed trajectory* $T_{\mathbf{x}}$ by

$$T_{\mathbf{x}} := \overline{\{\phi(\mathbf{x}, t) : t \in \mathbb{R}\}}. \tag{2}$$

Definition 2 (*Integral Set*)**.** Define the sets $I^i_{\mathbf{x}}$ for $i \in \mathbb{N}$ inductively by

$$I^1_{\mathbf{x}} = T_{\mathbf{x}},$$
$$I^i_{\mathbf{x}} = \left\{ \mathbf{y} \in \Omega : T_{\mathbf{y}} \cap I^{i-1}_{\mathbf{x}} \neq \emptyset \right\} \quad (i = 2, 3, 4, \ldots),$$

* Corresponding author at: School of Mathematical Sciences, University of Adelaide, Adelaide SA 5005, Australia.
E-mail address: sanjeevabalasuriya@yahoo.com (S. Balasuriya).

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and then the *integral set* $I_{\mathbf{x}}$ of $\mathbf{x}$ by

$$I_{\mathbf{x}} = \bigcup_{i \in \mathbb{N}} I_{\mathbf{x}}^{i}. \tag{3}$$

Thus, $I_{\mathbf{x}}$ includes all trajectories that can be connected to $\mathbf{x}$ by going through a countable number of "end states", such as fixed points or periodic orbits; connecting through an end state at infinity is not allowed. The set $I_{\mathbf{x}}$ may be either one- or two-dimensional, depending on the topology of Ω and the dynamics of (1). We will be concerned with pairs of points $\mathbf{a}$ and $\mathbf{b}$ with the property that any curve joining $\mathbf{a}$ to $\mathbf{b}$ must have nonzero absolute flux across it. Thus the integral sets through $\mathbf{a}$ and $\mathbf{b}$ should neither intersect, nor be connectable via a curve of fixed points:

Hypothesis 1.

(i) The integral sets $I_{\mathbf{a}}$ and $I_{\mathbf{b}}$ do not intersect, that is $I_{\mathbf{a}} \cap I_{\mathbf{b}} = \emptyset$, and
(ii) There does not exist a curve $\tilde{\mathcal{C}} \subseteq \Omega$ such that $\mathbf{f}(\mathbf{x}) = \mathbf{0}$ for all $\mathbf{x} \in \tilde{\mathcal{C}}$, and for which $I_{\mathbf{a}} \cap \tilde{\mathcal{C}}$ and $I_{\mathbf{b}} \cap \tilde{\mathcal{C}}$ are both nonempty.

We will define flux in relation to a non-negative time-varying weight function g. The basic example to keep in mind is that $g(\mathbf{x}, t)$ is a chemical concentration; the flux definitions will then compute the flux of the chemical (in chemical mass per unit time, say), at each instance in time. Note that we allow g to vary with time; this enables the flux computations even when the chemical concentration is not in equilibrium. As will be argued later, our definition for g will allow the modelling of more general situations, such as the flux of vorticity or temperature. Our weight function is in general defined by:

Definition 3 (*Weight Function*)**.** The *weight function* $g : \Omega \times \mathbb{R} \to [0, \infty)$ is such that $g(\cdot, t) \in C^1(\Omega)$ for any $t \in \mathbb{R}$.

At a fixed time t, the idea is to determine curves which extremise the flux; these curves are restricted to C^1 curves in Ω taking the form $\mathcal{C} = \{\mathbf{r}(p) : 0 \le p \le 1\}$. The following definitions for the flux are at a specific time instance t, and are therefore *instantaneous* in nature.

Definition 4 (*Weighted Local (Point) Flux*)**.** The *weighted local flux* (or *weighted point flux*) at differentiable points $\mathbf{r}(p)$ on a piecewise C^1 curve $\mathcal{C} = \{\mathbf{r}(p) : 0 \le p \le 1\} \subset \Omega$ at a time instance t is given by

$$L_g(\mathbf{r}(p), t) := g(\mathbf{r}(p), t)\, \mathbf{f}(\mathbf{r}(p)) \cdot J\mathbf{r}'(p), \tag{4}$$

where $J := \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$.

Since $J\mathbf{r}'(p)$ represents the leftwards-pointing normal direction when traversing the curve in the direction of increasing p, the local flux measures the strength of $g\mathbf{f}$ in the leftwards normal direction to the curve, weighted according to g. Integrating the local flux over a curve $\mathcal{C}$ gives the "weighted flux" across $\mathcal{C}$, in the following senses:

Definition 5 (*Weighted Signed Flux*)**.** The *weighted signed flux* $F_g^s(\mathcal{C}, t)$ across $\mathcal{C}$ at a time instance t is defined by

$$F_g^s(\mathcal{C}, t) := \int_0^1 \mathbf{f}(\mathbf{r}(p)) \cdot J \frac{\mathbf{r}'(p)}{|\mathbf{r}'(p)|} \left|\mathbf{r}'(p)\right| g(\mathbf{r}(p), t)\, \mathrm{d}p = \int_0^1 L_g(\mathbf{r}(p), t)\, \mathrm{d}p. \tag{5}$$

Definition 6 (*Weighted Absolute Flux*)**.** The *weighted absolute flux* $F_g^a(\mathcal{C}, t)$ across $\mathcal{C}$ at a time instance t is defined by

$$F_g^a(\mathcal{C}, t) := \int_0^1 \left| \mathbf{f}(\mathbf{r}(p)) \cdot J \frac{\mathbf{r}'(p)}{|\mathbf{r}'(p)|} \right| \left|\mathbf{r}'(p)\right| g(\mathbf{r}(p), t)\, \mathrm{d}p = \int_0^1 \left| L_g(\mathbf{r}(p), t) \right| \mathrm{d}p. \tag{6}$$

The "standard flux" (quantity of fluid crossing per unit time) is obtained by setting $g \equiv 1$, while more general g can be used to represent the flux associated with a passively transported chemical concentration or of a (passive or active) scalar field. To state our main result, we also need the following definitions.

Definition 7 (*Weighted Compressibility*)**.** The *weighted compressibility* function $\kappa_g : \Omega \times \mathbb{R} \to \mathbb{R}$ is defined by

$$\kappa_g(\mathbf{x}, t) := \nabla \cdot (\mathbf{f}(\mathbf{x}) g(\mathbf{x}, t)), \tag{7}$$

where ∇ denotes the derivative with respect to $\mathbf{x}$.

If $g \equiv 1$, the weighted compressibility becomes the divergence of the vector field (the compressibility of the flow). That is, $\kappa_1 = \nabla \cdot \mathbf{f}$.

Definition 8 (*Flow Derivative*)**.** The *flow derivative* of a scalar field $h(\mathbf{x}, t)$ on $\Omega \times \mathbb{R}$ which is C^1 in Ω for each $t \in \mathbb{R}$ is given by

$$D_{\mathbf{f}} h(\mathbf{x}, t) := \mathbf{f}(\mathbf{x}) \cdot \nabla h(\mathbf{x}, t). \tag{8}$$