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Isometries of the unitary groups and Thompson isometries of the spaces of invertible positive elements in C^*-algebras

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ABSTRACT

We show that the existence of a surjective isometry (which is merely a distance preserving map) between the unitary groups of unital C^*-algebras implies the existence of a Jordan *-isomorphism between the algebras. In the case of von Neumann algebras we describe the structure of those isometries showing that any of them is extendible to a real linear Jordan *-isomorphism between the underlying algebras multiplied by a fixed unitary element. We present a result of similar spirit for the surjective Thompson isometries between the spaces of all invertible positive elements in unital C^*-algebras.

1. Introduction

The study of linear isometries between function spaces or operator algebras has a long history dating back to the early 1930's. For an excellent comprehensive treatment of related results we refer to the two volume set [5,6]. The most fundamental and classical results of that research area are the Banach–Stone theorem describing the structure of all linear surjective isometries between the Banach spaces of continuous functions on compact Hausdorff spaces and its noncommutative generalization, Kadison's theorem [13], which describes the structure of all linear surjective isometries between general unital C^*-algebras. One immediate consequence of those results is that if two C^*-algebras are isometrically isomorphic as Banach spaces, then they are isometrically isomorphic as Jordan *-algebras, too. This provides a good example of how nicely the different sides (in the present case the linear algebraic-geometrical structure and the full algebraic, more precisely, Jordan *-algebraic structure) of one complex mathematical object may be connected to or interact with each other. We mention another famous result of similar spirit which also concerns isometries. This is the celebrated Mazur–Ulam theorem stating that any surjective isometry between normed real linear spaces is automatically an affine transformation (hence equals a real linear surjective isometry followed by a translation). This means that if two normed real linear spaces are isometric as metric spaces, then they are isometrically isomorphic as normed linear spaces, too.

Recently, we have made attempts to extend the Mazur–Ulam theorem for more general noncommutative metrical structures, especially for metric groups. In [9] we have obtained a few general results which show that, under given conditions,

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the surjective isometries between certain substructures of metric groups necessarily have an algebraic property, namely they locally preserve the so-called inverted Jordan triple product $ba^{-1}b$. Applying those general results we have determined the surjective isometries of several nonlinear metrical structures (especially metric groups) of continuous functions and linear operators.

As the main motivation to our present investigations we mention that in the paper [10] we described the surjective isometries of the unitary group of a Hilbert space. Moreover, in the former paper [19] the second author determined the so-called Thompson isometries of the space of all positive definite operators on a Hilbert space. Those structures and spaces are of considerable importance due to the roles they play in several areas of algebra and analysis.

Our primary aim in this paper is to generalize the above mentioned results substantially, namely to extend them for the setting of general C^*-algebras. In what follows we show that unital C^*-algebras with isometric unitary groups are necessarily Jordan *-isomorphic. In the cases of general von Neumann algebras and commutative unital C^*-algebras we present the complete descriptions of those isometries. It turns out that they are closely related to Jordan *-isomorphisms of the underlying algebras. We obtain a similar result concerning the structure of surjective Thompson isometries between the sets of invertible positive elements of general unital C^*-algebras.

We emphasize that in this paper by an isometry we mean merely a distance preserving transformation, we do not assume that it respects an algebraic operation of any kind.

2. Norm isometries of unitary groups

In the paper [7] the first author studied surjective isometries between open subgroups of the general linear groups of unital semisimple commutative Banach algebras and proved that those transformations can uniquely be extended to isometric real linear algebra isomorphisms of the underlying algebras followed by a multiplication by a fixed element. He continued those investigations in [8] for noncommutative algebras. Using the results obtained there, in [11] the first author and K. Watanabe could give the complete description of surjective isometries between open subgroups of the general linear groups of C^*-algebras.

In what follows employing an approach very different from the one used in the above mentioned papers, we obtain structural results for the surjective isometries of certain important substructures of general linear groups. These are the unitary groups in unital C^*-algebras equipped with the usual norm and the so-called twisted subgroups of all invertible positive elements endowed with the Thompson metric. Here, by a twisted subgroup of a group G we mean a subset K of G which contains the unit of G and satisfies $yx^{-1}y \in K$ for any $x, y \in K$.

Based on our Mazur–Ulam type general results obtained in [9], we show below that formally the same description as in [11] is valid for the surjective isometries of the unitary groups of von Neumann algebras. In fact, more generally, we obtain that if the unitary groups of two unital C^*-algebras are isometric merely as metric spaces, then the underlying two algebras are (isometrically) isomorphic as Jordan *-algebras.

For our results we need the concept of Jordan isomorphisms between algebras as well as a few facts about them. If A, B are complex algebras, then a linear (real linear) map $J : A \to B$ is called a Jordan homomorphism (real linear Jordan homomorphism) if it satisfies $J(a^2) = J(a)^2$ for every $a \in A$ or, equivalently, if it satisfies $J(ab + ba) = J(a)J(b) + J(b)J(a)$ for any $a, b \in A$. Clearly, every homomorphism $\phi : A \to B$ (that is a linear map such that $\phi(ab) = \phi(a)\phi(b)$ holds for any $a, b \in A$) as well as every antihomomorphism $\psi : A \to B$ (that is a linear map $\psi : A \to B$ satisfying $\psi(ab) = \psi(b)\psi(a)$, $a, b \in A$) is a Jordan homomorphism. A Jordan *-homomorphism $J : A \to B$ between *-algebras A, B is a Jordan homomorphism which preserves the involution in the sense that $J(a^*) = J(a)^*$ holds for all $a \in A$. By a Jordan *-isomorphism we mean a bijective Jordan *-homomorphism.

In what follows the units of unital algebras will be denoted by 1. If A, B are unital algebras and $J : A \to B$ is a surjective Jordan homomorphism, then by the proof of Proposition 1.3 in [23] we have

(i) $J(1) = 1$;
(ii) $J(aba) = J(a)J(b)J(a)$, $a, b \in A$ and this implies that $J(a^n) = J(a)^n$ holds for every $a \in A$ and positive integer n;
(iii) for every invertible $a \in A$ we have that $J(a)$ is also invertible and $J(a^{-1}) = J(a)^{-1}$.

It then follows that any Jordan isomorphism between unital algebras preserves the spectrum of elements. Moreover, if A, B are unital *-algebras and $J : A \to B$ is a surjective Jordan *-homomorphism, then J maps the unitary group of A into the unitary group of B. In the case of C^*-algebras this easily implies that J is contractive due to the fact that any element of norm less than one is the arithmetic mean of unitaries (Kadison–Pedersen theorem [15]).

If A is a *-algebra, then the real linear subspace of its self-adjoint elements is denoted by A_s. If A is unital, by a symmetry in A we mean a self-adjoint unitary element (equivalently, a unitary whose square is the identity). Clearly, $s \in A$ is a symmetry if and only if it can be written as $s = 2p - 1$ with a projection (self-adjoint idempotent) $p \in A$.

We are now in a position to present and prove our first result which reads as follows.

Theorem 1. *Let A_j be a unital C^*-algebra and U_j its unitary group, $j = 1, 2$. Assume $\phi : U_1 \to U_2$ is a surjective isometry (with respect to the norms given on A_1, A_2). Then we have*

$$\phi(\exp(iA_{1s})) = \phi(1)\exp(iA_{2s}) \quad (1)$$