J. Math. Anal. Appl. 409 (2014) 180–188

Contents lists available at ScienceDirect

Journal of Mathematical Analysis and Applications

journal homepage: www.elsevier.com/locate/jmaa

Nonlinear maps preserving Jordan *-products

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ARTICLE INFO

Article history:
Received 10 November 2012
Available online 19 July 2013
Submitted by J.D.M. Wright

Keywords:
Maps preserving η-$*$-product
von Neumann algebras
Isomorphism

ABSTRACT

Let η be a non-zero scalar. In this paper, we investigate a bijective map ϕ between two von Neumann algebras, one of which has no central abelian projections, satisfying $\phi(AB+\eta BA^*)=\phi(A)\phi(B)+\eta\phi(B)\phi(A)^*$ for all A, B in the domain. It is showed that ϕ is a linear *-isomorphism if η is not real and ϕ is a sum of a linear *-isomorphism and a conjugate linear *-isomorphism if η is real.

1. Introduction

Let $\mathcal{A}$ be a *-algebra and η be a non-zero scalar. For $A, B \in \mathcal{A}$, define the Jordan η-$*$-product of A and B as $A\diamondsuit_\eta B = AB + \eta BA^*$. The Jordan (-1)-$*$-product, which is customarily called the skew product, was extensively studied because it naturally arises in the problem of representing quadratic functionals with sesquilinear functionals (see, for example, [9,8,10]) and in the problem of characterizing ideals (see, for example, [3,7]). We believe that the general Jordan η-$*$-product, particularly the Jordan 1-$*$-product and the Jordan i-$*$-product, is meaningful in some research topics.

A map ϕ between *-algebras $\mathcal{A}$ and $\mathcal{B}$ is said to preserve the Jordan η-$*$-product if $\phi(A\diamondsuit_\eta B) = \phi(A)\diamondsuit_\eta\phi(B)$ for all $A, B \in \mathcal{A}$. An and Hou in [1] proved that a bijective map preserving the Jordan (-1)-$*$-product between algebras of all bounded linear operators on Hilbert spaces is either a linear *-isomorphism or a conjugate linear *-isomorphism. This result was extended to factor von Neumann algebras by Cui and Li [4]. In [2], Bai and Du generalized the result of Cui and Li to more general von Neumann algebras, proving that a bijective map preserving the Jordan (-1)-$*$-product between von Neumann algebras without central abelian projections is a sum of a linear *-isomorphism and conjugate linear *-isomorphism. Recently, Li et al. in [5] consider maps which preserve the Jordan 1-$*$-product and proved that such a map between factor von Neumann algebras is either a linear *-isomorphism or a conjugate linear *-isomorphism. In this paper, we will completely describe maps preserving the Jordan η-$*$-product between von Neumann algebras without central abelian projections for all non-zero scalars η.

Let us fix some notations and terminologies. Throughout, algebras and spaces are over the complex number field $\mathbb{C}$. Suppose that $\mathcal{A}$ is a von Neumann algebra on a Hilbert space H. By $Z(\mathcal{A})$ denote the center of $\mathcal{A}$, that is, $Z(\mathcal{A}) = \{A \in \mathcal{A} : AB = BA \text{ for all } B \in \mathcal{A}\}$. A projection P is called a central abelian projection if $P \in Z(\mathcal{A})$ and $P\mathcal{A}P$ is abelian. For $A \in \mathcal{A}$, the central carrier of A, denoted by $\overline{A}$, is the smallest central projection P satisfying $PA = A$. It is not difficult to see that $\overline{A}$ is the projection onto the closed subspace spanned by $\{BAx : B \in \mathcal{A}, x \in H\}$. If Q is a projection in $\mathcal{A}$, then $\underline{Q}$, called the core of Q, is the biggest central projection P satisfying $P \le Q$. If $\underline{Q} = 0$, we call Q a core-free projection. It is easy to verify that $\underline{Q} = 0$ if and only if $\overline{I - Q} = I$, where I is the identity operator.

Lemma 1.1 ([6, Lemma 4]). *Let $\mathcal{A}$ be a von Neumann algebra without central abelian projections. Then there exists a projection P in $\mathcal{A}$ such that $\underline{P} = 0$ and $\overline{P} = I$.*

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http://dx.doi.org/10.1016/j.jmaa.2013.07.019

Lemma 1.2. *Let $\mathcal{A}$ be a von Neumann algebra on a Hilbert space H. Let A be an operator in $\mathcal{A}$ and P a projection with $\overline{P} = I$.*
(1) *If $ABP = 0$ for all $B \in \mathcal{A}$, then $A = 0$;*
(2) *If η is a non-zero scalar and $(PT(I-P))\diamondsuit_\eta A = 0$ for all $T \in \mathcal{A}$, then $A(I-P) = 0$.*

Proof. (1) This is easily seen from the fact that $\{BPx : x \in H\}$ is dense in H.
(2) By iT replacing T in $PT(I-P)A + \eta A(I-P)T^*P = 0$, we get $PT(I-P)A - \eta A(I-P)T^*P = 0$ and hence $A(I-P)T^*P = 0$ for all $T \in \mathcal{A}$. By (1), $A(I-P) = 0$.

2. Additivity

The main result in this section is as follows.

Theorem 2.1. *Let $\mathcal{A}$ be a von Neumann algebra without central abelian projections and $\mathcal{B}$ be a *-algebra. Let η be a non-zero scalar. Suppose that ϕ is a bijective map from $\mathcal{A}$ onto $\mathcal{B}$ which satisfies $\phi(A\diamondsuit_\eta B) = \phi(A)\diamondsuit_\eta\phi(B)$ for all $A, B \in \mathcal{A}$. Then ϕ is additive.*

Before the proof, we remark that the hypothesis "$\mathcal{A}$ containing no central abelian projections" is needed in the above theorem. For example, for $a, b \in \mathbb{R}$, define $\phi(a+bi) = 4(a^3 + b^3 i)$. Then ϕ is a bijection from $\mathbb{C}$ onto itself. It is not difficult to verify that ϕ preserves the Jordan 1-$*$-product and the Jordan (-1)-$*$-product. However it is obviously not additive.

Proof. First we give a key technique. Suppose that $A_1, A_2, \ldots, A_n$ and T are in $\mathcal{A}$ such that $\phi(T) = \sum_{i=1}^n \phi(A_i)$. Then for $S \in \mathcal{A}$, we have

$$\phi(S\diamondsuit_\eta T) = \phi(S)\diamondsuit_\eta\phi(T) = \sum_{i=1}^n \phi(S)\diamondsuit_\eta\phi(A_i) = \sum_{i=1}^n \phi(S\diamondsuit_\eta A_i) \tag{2.1}$$

and

$$\phi(T\diamondsuit_\eta S) = \phi(T)\diamondsuit_\eta\phi(S) = \sum_{i=1}^n \phi(A_i)\diamondsuit_\eta\phi(S) = \sum_{i=1}^n \phi(A_i\diamondsuit_\eta S). \tag{2.2}$$

Claim 1. $\phi(0) = 0$.

By the surjectivity of ϕ, we can find $A \in \mathcal{A}$ such that $\phi(A) = 0$. Then $\phi(0) = \phi(0\diamondsuit_\eta A) = \phi(0)\diamondsuit_\eta\phi(A) = \phi(0)\diamondsuit_\eta 0 = 0$.

In what follows, we fix a non-trivial projection P_1 in $\mathcal{A}$ and let $P_2 = I - P_1$. Set $\mathcal{A}_{ij} = P_i\mathcal{A}P_j$. Then $\mathcal{A} = \sum_{i,j=1}^2 \mathcal{A}_{ij}$. When we write A_{ij}, it indicates that $A_{ij} \in \mathcal{A}_{ij}$.

Claim 2. Let $A_{ii} \in \mathcal{A}_{ii}$, $i = 1, 2$. Then $\phi(A_{11} + A_{22}) = \phi(A_{11}) + \phi(A_{22})$.

By surjectivity, choose $T = \sum_{i,j=1}^2 T_{ij} \in \mathcal{A}$ such that $\phi(T) = \phi(A_{11}) + \phi(A_{22})$. For any $\lambda \in \mathbb{C}$, since $(\lambda P_1)\diamondsuit_\eta A_{22} = 0$, it follows from (2.1) and Claim 1 that $\phi((\lambda P_1)\diamondsuit_\eta T) = \phi((\lambda P_1)\diamondsuit_\eta A_{11})$. By the injectivity of ϕ, we get that $(\lambda P_1)\diamondsuit_\eta T = (\lambda P_1)\diamondsuit_\eta A_{11}$, i.e.,

$$(\lambda + \eta\bar{\lambda})T_{11} + \lambda T_{12} + \eta\bar{\lambda}T_{21} = (\lambda + \eta\bar{\lambda})A_{11}.$$

Now letting $\lambda \neq 0$ and $\lambda + \eta\bar{\lambda} \neq 0$, we get $T_{12} = T_{21} = 0$ and $T_{11} = A_{11}$.

Similarly, we can get $T_{22} = A_{22}$, proving the claim.

Claim 3. Let $A_{12} \in \mathcal{A}_{12}$ and $A_{21} \in \mathcal{A}_{21}$. Then $\phi(A_{12} + A_{21}) = \phi(A_{12}) + \phi(A_{21})$.

Choose $T = \sum_{i,j=1}^2 T_{ij} \in \mathcal{A}$ such that $\phi(T) = \phi(A_{12}) + \phi(A_{21})$. For any $\lambda \in \mathbb{C}$, since $(\eta\lambda P_1 - \bar{\lambda}P_2)\diamondsuit_\eta A_{12} = 0$, it follows from (2.1) that $\phi((\eta\lambda P_1 - \bar{\lambda}P_2)\diamondsuit_\eta T) = \phi((\eta\lambda P_1 - \bar{\lambda}P_2)\diamondsuit_\eta A_{21})$. Hence by the injectivity of ϕ, we have

$$(\eta\lambda + |\eta|^2\bar{\lambda})T_{11} - (\bar{\lambda} + \eta\lambda)T_{22} - (\bar{\lambda} - |\eta|^2\bar{\lambda})T_{21} = -(\bar{\lambda} - |\eta|^2\bar{\lambda})A_{21}$$

for all $\lambda \in \mathbb{C}$. From this, we get that $T_{11} = T_{22} = 0$.

Now since $A_{12}\diamondsuit_\eta P_1 = 0$, it follows from (2.2) that $\phi(T\diamondsuit_\eta P_1) = \phi(A_{21}\diamondsuit_\eta P_1)$. Hence $T_{21} + \eta T_{21}^* = A_{21} + \eta A_{21}^*$, and so $T_{21} = A_{21}$. Similarly, $T_{12} = A_{12}$, proving the claim.

Claim 4. For $i, j, k \in \{1, 2\}$, $i \neq j$, $A_{kk} \in \mathcal{A}_{kk}$, $A_{ij} \in \mathcal{A}_{ij}$, we have $\phi(A_{kk} + A_{ij}) = \phi(A_{kk}) + \phi(A_{ij})$.

We only prove the case $i = k = 1$ and $j = 2$; the proof of the other cases is similar. Now suppose that T in $\mathcal{A}$ is such that $\phi(T) = \phi(A_{11}) + \phi(A_{12})$. Since $(\lambda P_2)\diamondsuit_\eta A_{11} = 0$, it follows from (2.1) that $\phi((\lambda P_2)\diamondsuit_\eta T) = \phi((\lambda P_2)\diamondsuit_\eta A_{12})$ for all scalar λ. Hence $T_{22} = T_{21} = 0$ and $T_{12} = A_{12}$.

Since $(\eta\lambda P_1 - \bar{\lambda}P_2)\diamondsuit_\eta A_{12} = 0$, it follows that $\phi((\eta\lambda P_1 - \bar{\lambda}P_2)\diamondsuit_\eta T) = \phi((\eta\lambda P_1 - \bar{\lambda}P_2)\diamondsuit_\eta A_{11})$ for all scalar λ and hence $T_{11} = A_{11}$.

Claim 5. For $A_{11} \in \mathcal{A}_{11}$, $A_{22} \in \mathcal{A}_{22}$, $B_{12} \in \mathcal{A}_{12}$, $C_{21} \in \mathcal{A}_{21}$, we have

$$\phi(A_{11} + B_{12} + C_{21}) = \phi(A_{11}) + \phi(B_{12}) + \phi(C_{21}) \tag{2.3}$$

and

$$\phi(A_{22} + B_{12} + C_{21}) = \phi(A_{22}) + \phi(B_{12}) + \phi(C_{21}). \tag{2.4}$$