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Applications of some elliptic equations in Riemannian manifolds☆

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ABSTRACT

Let (M^{n+1}, g) be a compact Riemannian manifold with smooth boundary B and nonnegative Bakry–Emery Ricci curvature. In this paper, we use the solvability of some elliptic equations to prove some estimates of the weighted mean curvature and some related rigidity theorems. As their applications, we obtain some lower bound estimate of the first nonzero eigenvalue of the drifting Laplacian acting on functions on B and some corresponding rigidity theorems.

1. Introduction and main results

Let (M^{n+1}, g) be a compact Riemannian manifold with smooth boundary $B^n = \partial M$. Let dV and dA be the canonical measures on M and B respectively, V and A be the volume of M and the area of B. Given $f \in C^\infty(M)$, Reilly's formula [11] states that

$$\int_M ((\bar{\Delta} f)^2 - |\bar{\nabla}^2 f|^2 - \mathrm{Ric}(\bar{\nabla} f, \bar{\nabla} f))dV = \int_B (2(\Delta f)f_\nu + nH(f_\nu)^2 + \Pi(\nabla f, \nabla f))dA, \tag{1.1}$$

here $\bar{\nabla} f$, $\bar{\Delta} f$, $\bar{\nabla}^2 f$ being the gradient, the Laplacian and the Hessian of f on M, Ric the Ricci curvature of M, ∇f and Δf the gradient and the Laplacian of f in B, and $\Pi(X, Y) = g(\bar{\nabla}_X \nu, Y)$ for $\forall X, Y \in TB$ and $H = \frac{1}{n}\mathrm{tr}\,\Pi$ the second fundamental form and the mean curvature of B with respect to the outer unit normal ν on B.

Reilly [12] used the formula (1.1) to prove that if M has nonnegative Ricci curvature with convex boundary and $H \geq \frac{A}{(n+1)V}$, then M is isometric to an Euclidean ball. Later Ros [13] removed the condition of the convex boundary and obtains the same conclusion.

Recently Ma and Du [7] studied the drifting Laplacian operator $\bar{\Delta}_h = \bar{\Delta} - \bar{\nabla} h \cdot \bar{\nabla}$ for a smooth function h on M. This operator is self-adjoint operator with respect to the weighted measure $dV_h = e^{-h} dV$. They extended the above Reilly's formula and showed that

$$\int_M ((\bar{\Delta}_h f)^2 - |\bar{\nabla}^2 f|^2 - \mathrm{Ric}_h(\bar{\nabla} f, \bar{\nabla} f))dV_h = \int_B (2(\Delta_h f)f_\nu + nH_h(f_\nu)^2 + \Pi(\nabla f, \nabla f))dA_h \tag{1.2}$$

here $\mathrm{Ric}_h = \mathrm{Ric} + \bar{\nabla}^2 h$, $\Delta_h f = \Delta f - \nabla h \cdot \nabla f$, $H_h = H - \frac{1}{n} h_\nu$ and $dA_h = e^{-h} dA$. In [15], the second author proved the sharp gradient estimate for positive solution of the Laplacian with a general drift B, i.e. $\bar{\Delta} f - Bf = 0$, where B is a vector field.

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Using the inequalities of $|\bar{\nabla}^2 f|^2 \geq \frac{1}{n+1}(\bar{\Delta} f)^2$ and $\frac{1}{n+1}a^2 + \frac{1}{m-n-1}b^2 \geq \frac{1}{m}(a-b)^2$, we know that the Eq. (1.2) become the following inequality:

$$\int_M \frac{m-1}{m}((\bar{\Delta}_h f)^2 - \mathrm{Ric}_m(\bar{\nabla} f, \bar{\nabla} f))dV_h \geq \int_B (2(\Delta_h f)f_\nu + nH_h(f_\nu)^2 + \Pi(\nabla f, \nabla f))dA_h \tag{1.3}$$

here $\mathrm{Ric}_m = \mathrm{Ric}_h - \frac{1}{m-n-1}\bar{\nabla} h \bigotimes \bar{\nabla} h$, $m \geq n+1$, and $m = n+1$ if and only if h is a constant. This curvature tensor is called Bakry–Emery Ricci curvature (see [5]). With the help of this inequality, Ma and Du obtained the lower bound for the first eigenvalue of the drifting Laplacian on the compact manifold with positive Bakry–Emery Ricci curvature (see [7]). It states that if $\mathrm{Ric}_m \geq (m-1)K > 0, H_h \geq 0$ or $\Pi \geq 0$, then the first Neumann eigenvalue $\lambda_1^N(\bar{\Delta}_h) \geq mK$, or the first Dirichlet eigenvalue $\lambda_1^D(\bar{\Delta}_h) \geq mK$. This conclusion is a generalization of Reilly's [12] and Escobar's results [2]. Recently Li and Wei [6] proved that this result is sharp, i.e. if $\lambda_1^D(\bar{\Delta}_h) = mK$ or $\lambda_1^N(\bar{\Delta}_h) = mK$, then the manifold is isometric to a Euclidean hemisphere. They extended the rigidity theorem of Reilly [12] and Escobar [2]. From Theorem 5 in [10], we know that if $\mathrm{Ric}_m \geq (m-1)K > 0$, then M is compact and the diameter $\mathrm{diam}(M) \leq \frac{\pi}{\sqrt{K}}$. In [16], the second author proved that if $\mathrm{Ric}_m \geq (m-1)K > 0$ and $\mathrm{diam}(M) = \frac{\pi}{\sqrt{K}}$, then M is isometric to a Euclidean sphere of radius $\frac{1}{\sqrt{K}}$.

Based on the Reilly formula (1.1), Ros [13] showed an estimate of the mean curvature. Similarly using the Reilly type inequality (1.3) we may extend Ros's result to the manifold with nonnegative Bakry–Emery Ricci curvature and obtain an estimate of the weighted mean curvature. However in this paper we do not use the Reilly type formula but the divergence theorem to prove this result.

Theorem 1.1. *Let (M^{n+1}, g) be a compact Riemannian manifold with smooth boundary B and $\mathrm{Ric}_m \geq 0$. If the weighted mean curvature $H_h > 0$, then*

$$\int_B \frac{dA_h}{H_h} \geq \frac{mn}{m-1}V_h, \tag{1.4}$$

here $V_h = \int_M dV_h$ denotes the weighted volume of M. The equality holds if and only if M is isometric to an Euclidean ball and h is constant.

Let $A_h = \int_B dA_h$, if $H_h \geq \frac{(m-1)A_h}{mnV_h}$, then the equality in (1.4) holds. Thus we easily deduce the following rigidity theorem.

Corollary 1.2. *Let (M^{n+1}, g) be a compact Riemannian manifold with smooth boundary B and $\mathrm{Ric}_m \geq 0$. If the weighted mean curvature $H_h \geq \frac{(m-1)A_h}{mnV_h}$, then M is isometric to an Euclidean ball and h is constant.*

Remark 1. When h is a constant, the Bakry–Emery Ricci curvature and the weighted mean curvature become the classical Ricci curvature and the mean curvature respectively, and $m = n+1$. In this case, Corollary 1.2 is Ros's result [13].

Using the similar method we prove the following estimate of the weighted mean curvature and the related rigidity theorem. If h is a constant, then it is a Reilly's result in [12].

Theorem 1.3. *Let (M^{n+1}, g) be a compact Riemannian manifold with convex boundary B and $\mathrm{Ric}_m \geq 0$. Then*

$$\int_B H_h dA_h \leq \frac{(m-1)A_h^2}{mnV_h}. \tag{1.5}$$

The equality holds if and only if M is isometric to an Euclidean ball and h is constant.

Now we discuss some applications of Corollary 1.2. Firstly we prove that if the manifold has nonnegative Bakry–Emery Ricci curvature, then the critical point of the weighted isoperimetric functional is an Euclidean ball. Recently there have been many results about isoperimetric problems on the manifold with density, for example, see [1,3,4,8,9,14] and so on.

Theorem 1.4. *Let (M^{n+1}, g) be a compact Riemannian manifold with $\mathrm{Ric}_m \geq 0$, and let Ω be a compact domain in M with smooth boundary $\partial\Omega$. If Ω is a critical point of the weighted isoperimetric functional*

$$\Omega \to \frac{A_h(\partial\Omega)^m}{V_h(\Omega)^{m-1}},$$

then Ω is isometric to an Euclidean ball and h is constant.

In [17], Xia used the Reilly's formula (1.1) and Ros's result [13] to obtain a lower bound of the first nonzero eigenvalue $\lambda_1(\Delta)$ of the Laplacian acting on functions on B and the corresponding rigidity theorem.

Theorem 1.5 (*Xia's Theorem in [17]*)**.** *Let (M^{n+1}, g) be a compact Riemannian manifold with nonempty boundary B and non-negative Ricci curvature. If the second fundamental form of B satisfies $\Pi \geq cI$ (in the matrix sense), then $\lambda_1(\Delta) \geq nc^2$. The equality holds if and only if M is isometric to an Euclidean ball.*