J. Math. Anal. Appl. 409 (2014) 197–204

Contents lists available at ScienceDirect

Journal of Mathematical Analysis and Applications

journal homepage: www.elsevier.com/locate/jmaa

Secant zeta functions☆

Matilde Lalín *, Francis Rodrigue, Mathew Rogers

Department of Mathematics and Statistics, University of Montreal, Montreal, Canada

ARTICLE INFO

Article history:
Received 16 October 2012
Available online 9 July 2013
Submitted by B.C. Berndt

Keywords:
Secant zeta function
Bernoulli numbers
Clausen functions
Riemann zeta function

ABSTRACT

We study the series $\psi_s(z) := \sum_{n=1}^{\infty} \sec(n\pi z)n^{-s}$ and prove that it converges under mild restrictions on z and s. The function possesses a modular transformation property, which allows us to evaluate $\psi_s(z)$ explicitly at certain quadratic irrational values of z. This supports our conjecture that $\pi^{-k}\psi_k(\sqrt{j}) \in \mathbb{Q}$ whenever k and j are positive integers with k even. We conclude with some speculations on the Bernoulli numbers.

1. Introduction

Let $\zeta(s)$ denote the Riemann zeta function. It is well known that $\zeta(2n)\pi^{-2n} \in \mathbb{Q}$ for $n \geq 1$. Dirichlet L-functions and Clausen functions are modified versions of the Riemann zeta function, which also have nice properties at integer points [7]. Berndt studied a third interesting modification of the Riemann zeta function, namely the cotangent zeta function [1]:

$$\xi_s(z) := \sum_{n=1}^{\infty} \frac{\cot(\pi nz)}{n^s}. \tag{1.1}$$

He proved that (1.1) converges under mild restrictions on z and s, and he produced many explicit formulas for $\xi_k(z)$, when z is a quadratic irrational, and $k \geq 3$ is an odd integer. Consider the following examples:

$$\xi_3\left(\frac{1+\sqrt{5}}{2}\right) = -\frac{\pi^3}{45\sqrt{5}}, \qquad \xi_5(\sqrt{2}) = \frac{\pi^5}{945\sqrt{2}}.$$

Berndt's work implies that $\sqrt{j}\,\xi_k(\sqrt{j})\pi^{-k} \in \mathbb{Q}$ whenever j is a positive integer that is not a perfect square, and $k \geq 3$ is odd. A natural extension of that work is to replace $\cot(z)$ with one of the functions $\{\tan(z), \csc(z), \sec(z)\}$. We can settle the tangent and cosecant cases via elementary trigonometric identities:

$$\sum_{n=1}^{\infty} \frac{\tan(\pi nz)}{n^s} = \xi_s(z) - 2\xi_s(2z), \qquad \sum_{n=1}^{\infty} \frac{\csc(\pi nz)}{n^s} = \xi_s(z/2) - \xi_s(z),$$

but it is more challenging to understand the secant zeta function:

$$\psi_s(z) := \sum_{n=1}^{\infty} \frac{\sec(\pi nz)}{n^s}. \tag{1.2}$$

☆ This work has been partially supported by NSERC Discovery Grant 355412-2008 and FQRNT Subvention établissement de nouveaux chercheurs 144987. The work of FR has also been supported by a Bourse d'été de premier cycle du ISM-CRM.

* Corresponding author.
E-mail addresses: mlalin@dms.umontreal.ca (M. Lalín), rodriguefrancis@gmail.com (F. Rodrigue), mathewrogers@gmail.com (M. Rogers).

0022-247X/$ – see front matter © 2013 Elsevier Inc. All rights reserved.
http://dx.doi.org/10.1016/j.jmaa.2013.06.071

The main goal of this paper is to prove formulas for special values of $\psi_s(z)$. In Section 2, we prove that the sum converges absolutely if z is an irrational algebraic number and $s \geq 2$. In Section 4, we obtain results such as

$$\psi_2(\sqrt{2}) = -\frac{\pi^2}{3}, \qquad \psi_2(\sqrt{6}) = \frac{2\pi^2}{3}.$$

These types of formulas exist because $\psi_k(z)$ obeys a modular transformation which we prove in Section 3 (see Eq. (3.8)). Furthermore, based on numerical experiments, we have the following conjecture.

Conjecture 1. *Assume that k and j are positive integers, and that k is even. Then $\psi_k(\sqrt{j})\pi^{-k} \in \mathbb{Q}$.*

The results of Section 4 support this conjecture, even though there are still technical hurdles to constructing a complete proof. For instance, we prove that the conjecture holds for infinite subsequences of natural numbers. The rational numbers that appear are also interesting, and we speculate on their properties in the conclusion.

2. Convergence

Since $\sec(\pi z)$ has poles at the half-integers, it follows that $\psi_s(z)$ is only well-defined if $nz \notin \mathbb{Z} + \frac{1}{2}$ for any integer n. Thus, we exclude rational points with even denominators from the domain of $\psi_s(z)$. If $z = p/q$ with q odd, then $\psi_s(p/q)$ reduces to linear combination of values of the Hurwitz zeta function, and (1.2) converges for $s > 1$. Convergence questions become more complicated if z is irrational. Irrationality guarantees that $|\sec(\pi nz)| \neq \infty$, but we still have to account for how often $|\sec(\pi nz)|$ is large compared to n^s. The Thue–Siegel–Roth theorem gives that $|\sec(\pi nz)| \ll n^{1+\varepsilon}$ when z is algebraic and irrational, and this proves that (1.2) converges for $s > 2$. The case when $s = 2$ requires a more subtle argument. We use a theorem of Worley to show that the set of n's where $|\sec(\pi nz)|$ is large is sparse enough to ensure that (1.2) converges. We are grateful to Florian Luca for providing this part of the proof. In summary, we have the following theorem.

Theorem 1. *The series in (1.2) converges absolutely in the following cases:*

(1) *When $z = p/q$ with q odd and $s > 1$.*
(2) *When z is algebraic irrational, and $s > 2$.*
(3) *When z is algebraic irrational, and $s = 2$.*

Proof of Theorem 1, parts (1) and (2). Let z be a rational number with odd denominator in reduced form. It is easy to see that the set of real numbers $\{\sec(n\pi z)\}_{n\in\mathbb{N}}$ is finite. Let $M = \max_{n\in\mathbb{N}} |\sec(n\pi z)|$. Then we have

$$\frac{|\sec(\pi nz)|}{n^s} \leq \frac{M}{n^s}.$$

It follows easily from the Weierstrass M-test that (1.2) converges absolutely for $s > 1$.

Now we prove the second part of the theorem. By elementary estimates

$$|\sec(\pi nz)| = |\csc(\pi(nz - 1/2))| \ll \frac{1}{\left|nz - \frac{1}{2} - k_n\right|}, \tag{2.1}$$

where k_n is the integer which minimizes $|nz - \frac{1}{2} - k_n|$. Now appeal to the Thue–Siegel–Roth theorem [11]. In particular, for any algebraic irrational number α, and given $\varepsilon > 0$, there exists a constant $C(\alpha, \varepsilon)$, such that

$$\left|\alpha - \frac{p}{q}\right| > \frac{C(\alpha, \varepsilon)}{q^{2+\varepsilon}}. \tag{2.2}$$

If we set $\alpha = z$, then (2.1) becomes

$$|\sec(\pi nz)| \ll \frac{1}{n\left|z - \frac{2k_n+1}{2n}\right|} \ll n^{1+\varepsilon}.$$

Therefore we have

$$\frac{|\sec(n\pi z)|}{n^s} \ll \frac{1}{n^{s-1-\varepsilon}},$$

and this implies that (1.2) converges absolutely for $s > 2 + \varepsilon$. Since ε is arbitrarily small the result follows. □

In order to prove the third part of Theorem 1, we require some background on continued fractions. Recall that any irrational number z can be represented as an infinite continued fraction

$$z = [a_0; a_1, a_2, \ldots],$$