J. Math. Anal. Appl. 409 (2014) 228–235

Contents lists available at ScienceDirect

Journal of Mathematical Analysis and Applications

journal homepage: www.elsevier.com/locate/jmaa

Hankel matrices acting on Dirichlet spaces

Guanlong Bao, Hasi Wulan *

Department of Mathematics, Shantou University, Shantou 515063, China

ARTICLE INFO

Article history:
Received 4 July 2012
Available online 9 July 2013
Submitted by D. Khavinson

Keywords:
Hankel matrix
Hilbert operator
Dirichlet spaces
Carleson measure

ABSTRACT

We give a connection between the Hankel matrix acting on Dirichlet spaces $\mathcal{D}_\alpha, 0 < \alpha < 2$, and the Carleson measure supported on $(-1, 1)$. As an application, we prove that the generalized Hilbert operators $\mathcal{H}_\beta$ are always bounded on Dirichlet spaces $\mathcal{D}_\alpha$ for $0 < \alpha < 2$ and that the range $(0, 2)$ of α in our results is the best possible.

1. Introduction

Let $\mathbb{D}$ be the open unit disk in the complex plane. Denote by $H(\mathbb{D})$ the class of functions analytic in $\mathbb{D}$. For $f \in H(\mathbb{D})$ and $0 < r < 1$, the integral mean $M_p(r, f)$ is defined by

$$M_p(r, f) = \left\{ \frac{1}{2\pi} \int_0^{2\pi} \left| f(re^{i\theta}) \right|^p d\theta \right\}^{\frac{1}{p}}.$$

The Hardy space H^p, $0 < p < \infty$, is the class of all $f \in H(\mathbb{D})$ for which

$$\|f\|_{H^p} = \sup_{0<r<1} M_p(r, f) < \infty.$$

The Dirichlet space $\mathcal{D}_\alpha$, $\alpha \in \mathbb{R}$, consists of functions $f(z) = \sum_{n=0}^{\infty} a_n z^n \in H(\mathbb{D})$ for which

$$\|f\|_{\mathcal{D}_\alpha}^2 = \sum_{n=0}^{\infty} (n+1)^{1-\alpha} |a_n|^2 < \infty.$$

For $\alpha = 0$ we obtain the classical Dirichlet space $\mathcal{D} = \mathcal{D}_0$, and for $\alpha = 1$ we get the Hardy space $H^2 = \mathcal{D}_1$. See [5,9,20].

A Hankel matrix, finite or infinite, is a matrix whose j, k entry is a function of $j+k$. Let μ be a finite positive Borel measure on $\mathbb{D}$. In [19], Widom considered the Hankel matrix $S[\mu] = (\mu[i+j])_{i,j \geq 0}$ with

$$\mu[i+j] = \int_{\mathbb{D}} z^{i+j} d\mu(z).$$

* Corresponding author.
E-mail addresses: glbaoah@163.com (G. Bao), wulan@stu.edu.cn (H. Wulan).

0022-247X/$ – see front matter
http://dx.doi.org/10.1016/j.jmaa.2013.07.006

The Hankel matrix $S[\mu]$ induces formally an operator, denoted also by $S[\mu]$, on $H(\mathbb{D})$ in the following sense. For $f(z) = \sum_{n=0}^{\infty} a_n z^n \in H(\mathbb{D})$, by multiplication of the matrix with the sequence of Taylor coefficients of the function f

$$\{a_n\}_{n\geq 0} \longmapsto \left\{\sum_{k=0}^{\infty} \mu[n+k]a_k\right\}_{n\geq 0},$$

define

$$S[\mu](f)(z) = \sum_{n=0}^{\infty}\left(\sum_{k=0}^{\infty} \mu[n+k]a_k\right)z^n.$$

Based on the results in [19], Power [17] built a connection between the Carleson measure and the Hankel matrix as follows.

Theorem A. *Let μ be a finite positive Borel measure on $\mathbb{D}$ supported on $(-1, 1)$.*

(i) *The following are equivalent.*
 (1) *μ is a Carleson measure.*
 (2) *$\mu[n] = O(n^{-1})$.*
 (3) *$S[\mu]$ is bounded on H^2.*

(ii) *The following are equivalent.*
 (1) *μ is a vanishing Carleson measure.*
 (2) *$\mu[n] = o(n^{-1})$.*
 (3) *$S[\mu]$ is compact on H^2.*

Galanopoulos and Peláez [12] characterized those positive Borel measures μ supported on $[0, 1)$ for which $S[\mu]$ is bounded or compact on H^1. Diamantopoulos, in [7], gave a series of results about the operators induced by Hankel matrices on Dirichlet spaces.

In this article, we show that the Hankel matrix acting on Dirichlet spaces $\mathcal{D}_\alpha$ can be characterized by a Carleson measure. This means that, like Theorem A, p-Carleson measures supported on $(-1, 1)$, $0 < p < \infty$, can also be characterized in terms of the related Hankel matrices. As an application, we prove that the generalized Hilbert operators $\mathcal{H}_\beta$ are always bounded on Dirichlet spaces $\mathcal{D}_\alpha$ for $0 < \alpha < 2$, and the range $(0, 2)$ of α is the best possible.

Let us recall the definition of the Carleson measure, which is a very useful tool in the study of Banach spaces of analytic functions. For $0 < p < \infty$, a positive Borel measure μ on $\mathbb{D}$ is a p-Carleson measure if

$$\sup_I \frac{\mu(S(I))}{|I|^p} < \infty$$

for all Carleson boxes

$$S(I) = \Big\{z \in \mathbb{D} : 1 - |I| < |z| < 1, |\arg z - \theta_I| < |I|\Big\},$$

where $|I|$ denotes the length of the arc I on $\mathbb{D}$ and θ_I is the midpoint of I. If $|I| \geq 1$, we note that $S(I) = \mathbb{D}$. If

$$\frac{\mu(S(I))}{|I|^p} \to 0$$

when $|I| \to 0$, we call μ the vanishing p-Carleson measure. See [2,3,10,17]. When $p = 1$, we get the classical (vanishing) Carleson measure. We refer to [1,4,14,18] for further results about Carleson measures.

In addition, the symbol $A \approx B$ means that $A \lesssim B \lesssim A$. We say that $A \lesssim B$ if there exists a constant C (independent of A and B) such that $A \leq CB$.

2. Main results

For $0 < p < \infty$, we define the Hankel matrix $S_p[\mu]$:

$$(S_p[\mu])_{i,j} = \int_{\mathbb{D}} (i+j+1)^{p-1} z^{i+j} d\mu(z), \quad i, j = 0, 1, 2, \ldots,$$

and an operator

$$S_p[\mu](f)(z) = \sum_{n=0}^{\infty}\left(\sum_{k=0}^{\infty}(n+k+1)^{p-1}\mu[n+k]a_k\right)z^n$$

for $f(z) = \sum_{n=0}^{\infty} a_n z^n \in H(\mathbb{D})$.

Now, we state one of our main results, which is the generalization of Theorem A.