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Semilinear elliptic problems with asymmetric nonlinearities☆

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ABSTRACT

In this paper we are concerned on the semilinear elliptic problem

$$\begin{cases} -\Delta u = -\lambda |u|^{q-2}u + au + b(u^+)^{p-1} & \text{in } \Omega, \\ u = 0, & \text{on } \partial\Omega, \end{cases}$$

where $\Omega \subseteq \mathbb{R}^N$ is a bounded domain with regular boundary $\partial\Omega$, $1 < q < 2 < p \leq 2^*$. If a is between two eigenvalues, we get the existence of three nontrivial solutions for the problem above.

1. Introduction

We consider the semilinear elliptic problem

$$\begin{cases} -\Delta u = -\lambda |u|^{q-2}u + au + b(u^+)^{p-1} & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{P}$$

where $\Omega \subseteq \mathbb{R}^N$ is a bounded domain with regular boundary $\partial\Omega$, $N \geq 3$, $1 < q < 2 < p \leq 2^*$, $a \in \mathbb{R}$, $b > 0$, λ is a positive parameter and $u^+ = \max\{u, 0\}$.

The weak solutions of the problem (P) correspond to critical points of the C^1 functional I_λ, defined on $H_0^1 :\equiv H_0^1(\Omega)$ by

$$I_\lambda(u) = \frac{1}{2}\int |\nabla u|^2 + \frac{\lambda}{q}\int |u|^q - \frac{a}{2}\int u^2 - \frac{b}{p}\int (u^+)^p, \quad u \in H_0^1. \tag{1}$$

After the appearance of [1], there has been increasing concern about multiple solutions of semilinear elliptic problem of the type:

$$-\Delta u = \mu |u|^{q-2}u + g(u) \quad \text{in } \Omega. \tag{2}$$

When g is asymmetric and asymptotically linear this problem was considered in [8,9,13,18]. Here asymmetric means that g satisfies an Ambrosetti–Prodi type condition (i.e. $g_- := \lim_{t\to-\infty} g(t)/t < \lambda_k < g_+ := \lim_{t\to+\infty} g(t)/t$). When g is asymmetric and superlinear at $+\infty$, $g_+ = \infty$, this problem was approached in [8,13,15]. In [8] a Neumann problem was considered and in [15] the authors studied a problem involving the p-Laplace operator. In [13], one was assumed that $g(t)/t$ crosses an eigenvalue of the Laplacian when the t varies from 0 to $-\infty$ (i.e. $g'(0) < \lambda_k < g_-$). Similar hypotheses also appear in [18]. Assumptions involving the first eigenvalue, as $g'(0), g_- \leq \lambda_1$, were considered in [8,9,15]. It is known that crossing eigenvalues, in particular the first one, is related to existence and multiplicity for such problems. Notice that the

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nonlinearity $g(t) = at + b(t^+)^{p-1}$, with $a > \lambda_1$, is not included in the cases count in the previous works. Moreover, similar problems with $\mu = 0$ were studied in [16] for Dirichlet problems, and in [2,17] for Neumann problems.

Our problem is also closely related to the class of superlinear Ambrosetti–Prodi problems:

$$-\Delta u = au + (u^+)^p + f(x) \quad \text{in } \Omega, \tag{3}$$

with $f \in L^2$. For instance, this problem has a solution if $\|f\|_{L^2}$ is small enough (see [11]). Further results and references for the above problem can be found in [5,6,10,12,19,20].

For the critical case, our main motivation to (P) is the Brezis–Nirenberg pioneering work [3], where the following critical problem was considered

$$\begin{cases} -\Delta u = au + |u|^{2^*-2}u & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases}$$

where $a < \lambda_1$. They noticed that the problem had a breaking of compactness at the value $\frac{S^{N/2}}{N}$, so that they constructed minimax levels for the energy functional associated below this value. Such ideas have been permeating many later works as well as ours. One of them was the Capozzi, Fortunato and Palmieri work [7]. They basically studied the problem above with a between two eigenvalues. They showed that the problem above has a nontrivial solution for all $a > 0$ when $N \geq 5$ and for a different from eigenvalues when $N = 4$.

We are denoting by $0 < \lambda_1 < \lambda_2 \leq \cdots \leq \lambda_j \leq \cdots$ the eigenvalues of $(-\Delta, H_0^1(\Omega))$ and by φ_j the correspondent eigenfunctions. The $H_0^1(\Omega)$ norm and $L^p(\Omega)$ norm are represented by $\|\cdot\|$ and $|\cdot|_p$ and we denote these spaces by H_0^1 and L^p, for simplicity, respectively.

In the following, we set up precisely the results obtained.

Theorem 1. *Let $N \geq 3$ and $\lambda_k < a < \lambda_{k+1}$. If $2 < p < 2^*$, then, for λ small enough, (P) has at least three nontrivial solutions.*

Theorem 2. *Let $N \geq 4$ and $\lambda_k < a < \lambda_{k+1}$. If $p = 2^*$, then, for λ small enough, (P) has at least three nontrivial solutions.*

The major arguments of the proofs of our theorems are based on variational methods. As it is well-known, we have to show some geometric conditions and prove a compactness condition. Provided with these tools, we obtain a negative and a positive solution and the third one comes from the linking theorem. In order to do that, we follow some tricks used in [6,14]. In the next section, we show the (PS) condition for the energy functional. In the third section, we present the proofs of the theorems above.

2. The (PS) condition

We begin by showing the (PS) condition for I_λ.

Lemma 1. *Let $\lambda_1 < a$, $2 < p \leq 2^*$ and $\lambda > 0$. Then every (PS) sequence of I_λ is bounded.*

Proof. Let (u_n) be a (PS) sequence for I_λ, i.e., it satisfies

$$\left| \frac{1}{2} \int |\nabla u_n| + \frac{\lambda}{q} \int |u_n|^q - \frac{a}{2} \int u_n{}^2 - \frac{b}{p} \int (u_n^+)^p \right| \leq C, \tag{4}$$

$$\left| \int \nabla u_n \nabla h + \lambda \int |u_n|^{q-2} u_n h - a \int u_n h - b \int (u_n^+)^{p-1} h \right| \leq \epsilon_n \|h\|, \quad \forall h \in H_0^1, \tag{5}$$

where $\epsilon_n \to 0$ as $n \to \infty$. By (4) and (5) we have

$$\begin{aligned} C + \epsilon_n \|u_n\| &\geq \left| I_\lambda(u_n) - \frac{1}{2} \langle I'_\lambda(u_n), u_n \rangle \right| \\ &= \left| \left(\frac{\lambda}{q} - \frac{\lambda}{2} \right) \int |u_n|^q + \left(\frac{b}{2} - \frac{b}{p} \right) \int (u_n^+)^p \right| \\ &\geq \left(\frac{b}{2} - \frac{b}{p} \right) \int (u_n^+)^p. \end{aligned}$$

Since $p > 2$ we get

$$\int (u_n^+)^p \leq C + \epsilon_n \|u_n\|. \tag{6}$$

We also have by (5),

$$|\langle I'_\lambda(u_n), u_n^- \rangle| = \left| \|u_n^-\|^2 + \lambda |u_n^-|_q^q - a|u_n^-|_2^2 \right| \leq \epsilon_n \|u_n^-\|, \tag{7}$$