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Space-like loxodromes on rotational surfaces in Minkowski 3-space

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ABSTRACT

We find all space-like loxodromes on rotational surfaces which have space-like meridians or time-like meridians, respectively by using a relevant Lorentzian angle in Minkowski 3-space. To understand loxodromes better, we draw some pictures of them via Mathematica computer program.

1. Introduction

Loxodromes (also known as rhumb lines) correspond to the curves which intersect all of the meridians at a constant angle on the Earth. An aircraft flying and a ship sailing on a fixed magnetic compass course move along a curve. Here the course is a rhumb and the curve is a loxodrome. Generally, a loxodrome is not a great circle, thus it does not measure the shortest distance between two points on the Earth. However loxodromes are important in navigation and they should be known by aircraft pilots and sailors [1].

A circle of constant latitude is a loxodrome since it is perpendicular to meridians. If we expect circles of latitude, all of the loxodromes on the sphere look like spherical spirals [1]. Also, surprisingly, loxodromes on the sphere are projected as logarithmic spirals and straight lines by using stereographic and Mercator projections, respectively [2,12].

On an arbitrary rotational surface, the meridians are copies of the profile curves. Briefly, in Euclidean 3-space, if we rotate the profile curve $\beta(s) = (f(s), 0, g(s)), f(s) > 0$ around the x_3-axis, the rotational surface is parametrized by $X(s,\theta) = (f(s)\cos\theta, f(s)\sin\theta, g(s))$. On this surface, the meridians are the $\theta =$ constant parameter curves. Thus the loxodromes are parametrized by $\alpha(s) = (f(s)\cos(\theta(s)), f(s)\sin(\theta(s)), g(s))$, where $\theta(s) = \int \frac{\sigma(s)}{f(s)}\, ds$ and $\sigma(s) = \sqrt{f'^2(s) + g'^2(s)}$ [2,8]. In Fig. 1, we can see a rotational surface (a sphere) and a loxodrome (a spherical spiral) on it.

Similar problems can be investigated in Minkowski 3-space. Like the screw motion in Euclidean 3-space, the Lorentzian screw motion which is a generalization of a rotation can be defined in Minkowski 3-space. Helicoidal surfaces under the Lorentzian screw motion have been studied by many authors, for example [5] and references therein. Also, in [2] Babaarslan and Munteanu have obtained different types of rotational surfaces in detail. Afterwards, they have computed all of the time-like loxodromes on the rotational surfaces in Minkowski 3-space.

Originally, Minkowski space derives from the relativity theory in Mathematical Physics. We give Lorentzian geometry in mathematics as an example. By using an invariant treatment of Lorentzian geometry, the relativity theory can be expressed

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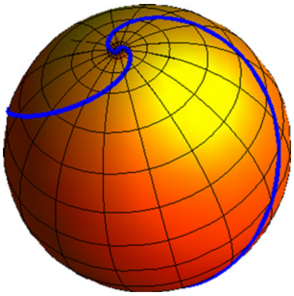

Fig. 1. Loxodrome on a sphere. Of course, only a part of the whole loxodrome is relevant for navigation.

in terms of Lorentzian geometry. Thus, surprisingly, geometers can enter into cosmology (i.e., redshift, expanding universe and big bang) and the gravitation of a single star (i.e., perihelion on procession, bending of light and black holes) [3,9].

Consequently, loxodromes in Minkowski 3-space have important meaning in relativity theory and they are interesting from the points of view of geometric and mathematical cosmology. In the present paper, we find all of the space-like loxodromes on the rotational surfaces which have either space-like meridians or time-like meridians, respectively by using a relevant Lorentzian angle in Minkowski 3-space.

2. Preliminaries

In this section, we briefly recall some important notions and also give different types of the rotational surfaces again in Minkowski 3-space. For more details, we refer to [2,4,6,9,10].

Let $\mathbb{R}^3_1$ be Minkowski 3-space, that is, $\mathbb{R}^3_1$ is real vector space $\mathbb{R}^3$ endowed with the Lorentzian metric

$$\langle u, v\rangle = u_1v_1 + u_2v_2 - u_3v_3, \tag{1}$$

where $u = (u_1, u_2, u_3)$ and $v = (v_1, v_2, v_3)$ are vectors in $\mathbb{R}^3$.

An arbitrary vector u in $\mathbb{R}^3_1$ is called;

i. space-like vector if $\langle u, u\rangle > 0$ or $u = 0$,
ii. time-like vector if $\langle u, u\rangle < 0$,
iii. light-like (null) vector if $\langle u, u\rangle = 0$ and $u \neq 0$.

This category of the given vector $u \in \mathbb{R}^3_1$ is called its causal character. The pseudo-norm (length) of the vector $u \in \mathbb{R}^3_1$ is $\|u\| = \sqrt{|\langle u, u\rangle|}$ and u is called a unitary vector if $\|u\| = 1$. Also, we say that two vectors $u, v \in \mathbb{R}^3_1$ are orthogonal if and only if $\langle u, v\rangle = 0$.

Let $\alpha : I \subset \mathbb{R} \to \mathbb{R}^3_1$ be a smooth (C^∞) regular curve in $\mathbb{R}^3_1$ (i.e., $\dot{\alpha}(t) \neq 0$ for all $t \in I$). α is called space-like, time-like or light-like curve if all of its velocity vectors $\dot{\alpha}(t)$ are space-like, time-like or light-like, respectively. This terminology derives from the relativity theory in Mathematical Physics. For instance, a space-like curve corresponds to the moving greater than the speed of light, a time-like curve corresponds to the moving smaller than the speed of light and a light-like curve corresponds to the moving which is certainly equal to the speed of light [3,7]. If α is space-like or time-like, then we say that α is a non-light-like curve. In this case, the arc-length of α, measured from $\alpha(t_0)$, $t_0 \in I$ is given by

$$s(t) = \int_{t_0}^{t} \|\dot{\alpha}(t)\| dt. \tag{2}$$

Then the parameter s is determined as $\|\alpha'(s)\| = 1$. Thus α is called a unit speed curve if $\|\alpha'(s)\| = 1$ [10].

In $\mathbb{R}^3_1$, there are three different types of rotations which leave an axis invariant. We can determine these rotations with the following rotational matrices:

The matrix corresponding to the rotation about space-like axis (i.e., x_1-axis) is

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & \cosh\theta & \sinh\theta \\ 0 & \sinh\theta & \cosh\theta \end{bmatrix}, \quad \theta \in \mathbb{R}. \tag{3}$$

The matrix corresponding to the rotation about time-like axis (i.e., x_3-axis) is

$$\begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad 0 \leq \theta \leq 2\pi. \tag{4}$$

This matrix seems like the Euclidean rotation matrix in shape.