J. Math. Anal. Appl. 409 (2014) 299–314

Contents lists available at ScienceDirect

Journal of Mathematical Analysis and Applications

journal homepage: www.elsevier.com/locate/jmaa

On Piterbarg's max-discretisation theorem for multivariate stationary Gaussian processes

Zhongquan Tan [a], Enkelejd Hashorva [b,*]

[a] College of Mathematics, Physics and Information Engineering, Jiaxing University, Jiaxing 314001, PR China

[b] Department of Actuarial Science, Faculty of Business and Economics, University of Lausanne, Switzerland

ARTICLE INFO

Article history:
Received 16 August 2012
Available online 18 July 2013
Submitted by Magda Peligrad

Keywords:
Berman condition
Strong dependence
Time discretisation
Piterbarg's max-discretisation theorem
Limit theorems
Multivariate stationary Gaussian processes

ABSTRACT

Let $\{X(t), t \geq 0\}$ be a stationary Gaussian process with zero-mean and unit variance. A deep result derived in Piterbarg (2004) [23], which we refer to as Piterbarg's max-discretisation theorem gives the joint asymptotic behaviour ($T \to \infty$) of the continuous time maximum $M(T) = \max_{t\in[0,T]} X(t)$, and the maximum $M^\delta(T) = \max_{t\in\mathfrak{R}(\delta)} X(t)$, with $\mathfrak{R}(\delta) \subset [0, T]$ a uniform grid of points of distance $\delta = \delta(T)$. Under some asymptotic restrictions on the correlation function Piterbarg's max-discretisation theorem shows that for the limit result it is important to know the speed $\delta(T)$ approaches 0 as $T \to \infty$. The present contribution derives the aforementioned theorem for multivariate stationary Gaussian processes.

1. Introduction

Let $\{X(t), t \geq 0\}$ be a standard (zero-mean, unit-variance) stationary Gaussian process with correlation function $r(\cdot)$ and continuous sample paths. A tractable and very large class of correlation functions satisfy

$$r(t) = 1 - C|t|^\alpha + o(|t|^\alpha) \quad \text{as } t \to 0 \tag{1}$$

for some positive constant C and $\alpha \in (0, 2]$, see e.g., [21]. If further, the Berman condition (see [2] or [3])

$$\lim_{T\to\infty} r(T) \ln T = 0 \tag{2}$$

holds, then it is well-known, see e.g., [16] that the maximum $M(T) = \max_{t\in[0,T]} X(t)$ obeys the Gumbel law as $T \to \infty$, namely

$$\lim_{T\to\infty} \sup_{x\in\mathbb{R}} \left| P\{a_T(M(T) - b_T) \leq x\} - \Lambda(x) \right| = 0 \tag{3}$$

is valid with $\Lambda(x) = \exp(-\exp(-x)), x \in \mathbb{R}$ the cumulative distribution function of a Gumbel random variable and normalising constants defined for all large T by

$$a_T = \sqrt{2\ln T}, \qquad b_T = a_T + a_T^{-1} \ln\left((2\pi)^{-1/2} C^{1/\alpha} H_\alpha a_T^{-1+2/\alpha}\right). \tag{4}$$

* Corresponding author.
E-mail address: Enkelejd.hashorva@unil.ch (E. Hashorva).

0022-247X/$ – see front matter
http://dx.doi.org/10.1016/j.jmaa.2013.07.022

Here H_α denotes the well-known Pickands constant given by the limit relation

$$H_\alpha = \lim_{S\to\infty} S^{-1}\mathbb{E}\left\{\exp\left(\max_{t\in[0,S]}\left(\sqrt{2}B_{\alpha/2}(t)-t^\alpha\right)\right)\right\} \in (0,\infty),$$

with B_α a standard fractional Brownian motion with Hurst index α, see e.g., [18] for recent characterisations of B_α. For the main properties of Pickands and related constants, see for example [1,21,4,5,7,30,6,8,9]. We note in passing that the first correct proof of the Pickands theorem where H_α appears (see [20]) is derived in [22].

We say that X is weakly dependent if its correlation function satisfies the Berman condition (2). A natural generalisation of (2) is the following assumption

$$\lim_{T\to\infty} r(T)\ln T = r \in (0,\infty) \tag{5}$$

in which case we say that X is a strongly dependent Gaussian process. [19] proves the limit theorem for the normalised maximum of strongly dependent stationary Gaussian processes showing that the limiting distribution is a mixture of Gumbel and Gaussian distributions. In fact, a similar result is shown therein also for the extreme case that (5) holds with $r=\infty$ with the limiting distribution being Gaussian. For other related results on extremes of strongly dependent Gaussian sequences and processes, we refer to [17,15,27,14,25,11,10], and the references therein.

In this paper $M(T) = \sup_{0\le t\le T} X(t), T > 0$ denotes the continuous-time maximum and $M^\delta(T) = \max_{t\in\delta\mathbb{N}\cap[0,T]} X(t)$ stands for the maximum over the uniform grid $\delta\mathbb{N}\cap[0,T]$. Under the assumption (1) we need to distinguish between three types of grids: a uniform grid of points $\mathfrak{R}(\delta)=\delta\mathbb{N}$ is called sparse if $\delta=\delta(T)$ is such that

$$\lim_{T\to\infty} \delta(T)(2\ln T)^{1/\alpha} = D, \tag{6}$$

with $D=\infty$. When (6) holds for some $D\in(0,\infty)$, then the grid is referred to as the Pickands grid, whereas when (6) holds with $D=0$, it is called a dense grid. Throughout this paper we assume that $\alpha\in(0,2]$.

[23] derived the joint asymptotic behaviour of $M^\delta(T)$ and $M(T)$ for weakly dependent stationary Gaussian processes. As shown therein, after a suitable normalisation (as in (3)) $M^\delta(T)$ and $M(T)$ are asymptotically independent, dependent or totally dependent if the grid is a sparse, a Pickands or a dense grid, respectively. We shall refer to that result as Piterbarg's max-discretisation theorem.

For a large class of locally stationary Gaussian processes [12] proved a similar result to [23] considering only sparse and dense grids. In another investigation concerning the storage process with fractional Brownian motion as input, it was shown in [13] that the continuous time maximum and the discrete time maximum over the dense grid are asymptotically completely dependent. [26,28] recently proved Piterbarg's max-discretisation theorem for strongly dependent stationary Gaussian processes, whereas [24] derives similar results for sparse and dense grids for standardised maximum of stationary Gaussian processes. Novel and deep results concerning stationary non-Gaussian processes are derived in [29].

As noted in [23] derivation of the joint asymptotic behaviour of $M^\delta(T)$ and $M(T)$ is important for theoretical problems and at the same time is crucial for applications, see [23,12,24] for more details.

The main contribution of this paper is the derivation of Piterbarg's max-discretisation theorem for multivariate stationary Gaussian processes. Our results show that, despite the high technical difficulties, it is possible to state Piterbarg's result in multidimensional settings allowing for asymptotic conditions and the two maxima are no longer asymptotically independent.

The brief organisation of the paper is as follows. In Section 2 we present the principal results. Section 3 presents some auxiliary results followed by Section 4 which is dedicated to the proofs of our main theorems. Several technical lemmas and the proof of Lemma 3.1 are displayed in the Appendix.

2. Main results

Consider $(X_1(t),\dots,X_p(t))$, $p\in\mathbb{N}$ a p-dimensional centred Gaussian vector process with covariance functions $r_{kk}(\tau)=\mathrm{Cov}(X_k(t),X_k(t+\tau))$, $k\le p$. Hereafter we shall assume that the components have continuous sample paths and further $\mathrm{Cov}(X_k(t),X_l(t+\tau))$ does not dependent on t so we shall write

$$r_{kl}(\tau)=\mathrm{Cov}(X_k(t),X_l(t+\tau))$$

for the cross-covariance function. Further we shall suppose that each component X_i has a unit variance function; in short we shall refer to such vector processes as standard stationary Gaussian vector processes. Similarly to (1) we suppose that for all indices $k\le p$

$$r_{kk}(t)=1-C|t|^\alpha+o(|t|^\alpha)\quad \text{as } t\to 0, \tag{7}$$

with some positive constants C, and further

$$\lim_{T\to\infty} r_{kl}(T)\ln T = r_{kl}\in(0,\infty) \tag{8}$$

holds for $1\le k,l\le p$. In order to exclude the possibility that $X_k(t)=\pm X_l(t+t_0)$ for some $k\ne l$, and some choice of t_0 and $+$ or $-$, we assume that