J. Math. Anal. Appl. 409 (2014) 315–320

Contents lists available at ScienceDirect

Journal of Mathematical Analysis and Applications

journal homepage: www.elsevier.com/locate/jmaa

Regularization by a modified quasi-boundary value method of the ill-posed problems for differential-operator equations of the first order

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ARTICLE INFO

Article history:
Received 9 March 2013
Available online 16 July 2013
Submitted by Barbara Kaltenbacher

Keywords:
Final value problem
Initial value problem
Ill-posed problem
Quasi-boundary value problem

ABSTRACT

In this paper, we consider the differential-operator equation

$$\frac{du\,(t)}{dt} + Au\,(t) = 0,$$

with A a self-adjoint unbounded operator coefficient, which does not have a fixed sign. The Cauchy problem for the equation above with conditions of the form

$$u(0) = f \quad \text{or} \quad u\,(T) = f,$$

is known to be an ill-posed problem. In this work, we will use a modified quasi-boundary value method; we obtain an approximate non-local problem depending on a small parameter $\alpha \in\,]0, 1[$. We show that the approximate problems are well-posed and that their solutions converge if the original problem has a classical solution. We also obtain a convergence result for these solutions.

1. Introduction

Let H be a Hilbert separable space, and let A be an unbounded self-adjoint operator in H with arbitrary sign, and which admits a compact inverse A^{-1}.

In this paper we consider the following problem:

Find a function $u : [0, T] \longrightarrow H$, that satisfies the equation

$$\frac{du(t)}{dt} + Au(t) = 0, \quad t \in [0, T]\,, \tag{1}$$

with the initial condition

$$u(0) = f, \tag{2}$$

or the final condition

$$u(T) = f, \tag{3}$$

where f is a given element in H.

Such forward Cauchy problems (1), (2) and backward Cauchy problems (1), (3) are ill-posed problems. Even though a unique solution in $[0, T]$ exists, it does not depend continuously on the value of f. We mention that the class of well-posed forward and ill-posed backward problems (corresponding in our case to A being of constant sign, positive or negative) has been treated by many authors, using many different approaches. Lavrentiev [5], Lattes and Lions [4], Miller [6], Payne [7]

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0022-247X/$ – see front matter
http://dx.doi.org/10.1016/j.jmaa.2013.07.013

and Showalter [8] have studied this problem by perturbing the operator A, and others, such as the authors of [3,1,9], used perturbation of the final value condition.

A similar problem is treated in a different way by N.I. Yurchuk and M. Ababneh [10]; they approximate the problems (1), (2) and (1), (3) by introducing into Eq. (1) the non-standard condition

$$\alpha u(0) + (1-\alpha)\, u(T) = f. \tag{4}$$

Also, we should mention that non-standard conditions of the form (4) for parabolic equations have been considered in [1].

In this work we approximate the problems (1), (2) and (1), (3) as follows:

Find the function $u : [0, T] \longrightarrow H$ that satisfies Eq. (1) and the non-local boundary condition

$$\alpha u(0) + (1-\alpha)\, u(T) + \alpha\,(1-\alpha)\left(u'(T) - u'(0)\right) = f, \tag{5}$$

where $\alpha \in\,]0, 1[$.

We see formally, when α tends towards 1 (resp. α tends towards 0), that the problem (1), (5) becomes the problem (1), (2) (resp. the problem (1), (3)).

Following [2,9], the related approximate problem is called the quasi-boundary value problem (QBVP). We show that the approximate problems (1), (5) are well-posed for each $\alpha \in\,]0, 1[$, and that their classical solutions u_α have the following properties:

$$\lim_{\alpha\to 0} \|u_\alpha(T) - f\| = 0; \qquad \lim_{\alpha\to 1} \|u_\alpha(0) - f\| = 0.$$

Furthermore, if the problem (1), (3) (resp. the problem (1), (2)) has a classical solution u_F (resp. u_I) then the sequence $\left(u'_\alpha(0)\right)_\alpha$ (resp. $\left(u'_\alpha(T)\right)_\alpha$) converges in H when $\alpha \to 0$ (resp. $\alpha \to 1$).

2. The approximate problem

In the following we study the problem (1), (5). We need a definition:

Definition 1. The function $u_\alpha : [0, T] \longrightarrow H$ is called the classical solution of the problem (1), (5) if $u_\alpha \in C^1([0, T], H)$, $u_\alpha(t) \in D(A)$, $\forall t \in [0, T]$, satisfies Eq. (1) and the condition (5).

Since we assumed that A admits an inverse A^{-1} compact, let us denote by $(\lambda_i)_{i\geq 1}$ the positive eigenvalues of A, by $\left(\mu_j\right)_{j\geq 1}$ the negative eigenvalues of A, by $(\varphi_i)_{i\geq 1}$ the eigenvectors of A corresponding to eigenvalues $(\lambda_i)_{i\geq 1}$, and by $\left(\psi_j\right)_{j\geq 1}$ the eigenvectors of A corresponding to eigenvalues $\left(\mu_j\right)_{j\geq 1}$.

The system $\left\{\left(\psi_j\right)_{j\geq 1}, (\varphi_i)_{i\geq 1}\right\}$ of the eigenvectors of the operator A form an orthogonal system in H; in addition we can assume that $\|\varphi_i\| = 1$ and $\left\|\psi_j\right\| = 1,\ \forall i, j \geq 1$.

Thus, for each $f \in H$, we can represent f in the form

$$f = \sum_{j=1}^{+\infty} a_j \psi_j + \sum_{i=1}^{+\infty} b_i \varphi_i, \tag{6}$$

where

$$a_j = \left(f, \psi_j\right), \qquad b_i = (f, \varphi_i), \quad \forall i, j \geq 1. \tag{7}$$

If the problem (1), (2) (resp. the problem (1), (3)) admits a solution u_I (resp. u_F), then this solution is represented in the form

$$u_I(t) = \sum_{j=1}^{+\infty} a_j e^{-\mu_j t} \psi_j + \sum_{i=1}^{+\infty} b_i e^{-\lambda_i t} \varphi_i, \quad \forall t \in [0, T], \tag{8}$$

$$u_F(t) = \sum_{j=1}^{+\infty} a_j e^{\mu_j (T-t)} \psi_j + \sum_{i=1}^{+\infty} b_i e^{\lambda_i (T-t)} \varphi_i, \quad \forall t \in [0, T]. \tag{9}$$

If the problem (1), (5) admits a solution u_α, then this solution is represented in the form

$$u_\alpha(t) = \sum_{j=1}^{+\infty} \beta_j a_j e^{\mu_j (T-t)} \psi_j + \sum_{i=1}^{+\infty} \gamma_i b_i e^{-\lambda_i t} \varphi_i, \quad \forall t \in [0, T], \tag{10}$$

where

$$\begin{aligned} \beta_j &= \left[\alpha e^{\mu_j T} + (1-\alpha) - \alpha\,(1-\alpha)\,\mu_j\left(1 - e^{\mu_j T}\right)\right]^{-1} \\ \gamma_i &= \left[\alpha + (1-\alpha)\, e^{-\lambda_i T} + \alpha\,(1-\alpha)\,\lambda_i\left(1 - e^{-\lambda_i T}\right)\right]^{-1}. \end{aligned} \tag{11}$$